5

Waves, Shocks, and Winds

Having derived the equations of fluid dynamics, we are now in a position to analyze some flows of interest. We shall confine attention to only a few illustrative examples of astrophysical importance. We consider first the propagation of small-amplitude disturbances in both homogeneous and stratified media, that is, *acoustic waves* and *acoustic-gravity waves* such as are observed in the solar atmosphere. Here it is adequate to use linearized equations of hydrodynamics. We then consider the nonlinear equations in the context of the generation, structure, propagation, and dissipation of *shocks*, which are important in a wide variety of astrophysical problems. Finally, we examine a nonlinear, radial, steady-flow problem that provides a first rough approximation to the physics of *stellar winds*, which are responsible for stellar mass-loss via supersonic flow into interstellar space.

5.1 Acoustic Waves

Acoustic waves are small-amplitude disturbances that propagate in a compressible medium through the interplay between fluid inertia and the restoring force of pressure. In order to isolate distinctly the characteristic properties of pure acoustic waves we assume that the medium is homogeneous, isotropic, and of infinite extent, and that no externally imposed forces act.

48. The Wave Equation

Take the ambient medium to be a perfect gas at rest with constant density ρ_0 and pressure p_0. Impose a small disturbance that perturbs these quantities locally to $\rho = \rho_0 + \rho_1$ and $p = p_0 + p_1$, where $|\rho_1/\rho_0| \ll 1$ and $|p_1/p_0| \ll 1$. The fluid acquires a small fluctuating velocity $\mathbf{v}_1$ such that $|\mathbf{v}_1|/a \ll 1$, where a is the speed of sound (see below). Velocity gradients are assumed to be so small that viscous effects are negligible, and the time scale for conductive heat transport is assumed to be so large compared to a characteristic fluctuation time that energy exchange by conduction can be ignored. In the absence of these dissipative processes, the wave-induced changes in gas properties are adiabatic, and because the undisturbed medium is homogeneous, the resulting flow is isentropic.

LINEARIZED FLUID EQUATIONS

Because all the perturbations ρ_1, p_1, and $\mathbf{v}_1$ are small, we can linearize the fluid equations, discarding all terms of second or higher order in these quantities. The equation of continuity (19.4) then becomes

$$(\partial \rho_1/\partial t) + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{48.1}$$

and Euler's equation (23.6) becomes

$$\rho_0(\partial \mathbf{v}_1/\partial t) + \nabla p_1 = 0. \tag{48.2}$$

We can dispense with the energy equation because the disturbance is adiabatic, which implies variations in material properties can be related through derivatives taken at constant entropy. In particular,

$$p_1 = (\partial p/\partial \rho)_s \rho_1. \tag{48.3}$$

Taking the curl of (48.2) we find

$$\partial(\nabla \times \mathbf{v}_1)/\partial t \equiv 0, \tag{48.4}$$

hence the vorticity $\boldsymbol{\omega}$ of the disturbed fluid remains constant in time. As the undisturbed fluid was initially at rest it follows that $\boldsymbol{\omega} \equiv 0$ at all times, and the disturbance is a potential flow with

$$\mathbf{v}_1 = \nabla \phi_1, \tag{48.5}$$

where ϕ_1 is the velocity potential.

THE WAVE EQUATION

Taking $(\partial/\partial t)$ of (48.1) and subtracting the divergence of (48.2) we find

$$(\partial^2 \rho_1/\partial t^2) - \nabla^2 p_1 = 0, \tag{48.6}$$

which, by virtue of (48.3), implies that

$$(\partial^2 \rho_1/\partial t^2) - a^2\, \nabla^2 \rho_1 = 0 \tag{48.7}$$

and

$$(\partial^2 p_1/\partial t^2) - a^2\, \nabla^2 p_1 = 0, \tag{48.8}$$

where

$$a^2 \equiv (\partial p/\partial \rho)_s. \tag{48.9}$$

Furthermore, using (48.5) and (48.9) in (48.2) we find

$$\rho_0[\partial(\nabla \phi_1)/\partial t] + a^2\, \nabla \rho_1 = \nabla[\rho_0(\partial \phi_1/\partial t) + a^2 \rho_1] = 0, \tag{48.10}$$

which implies

$$\rho_0(\partial \phi_1/\partial t) + a^2 \rho_1 = C \tag{48.11}$$

where C is a constant over all space. But both ϕ_1 and ρ_1 vanish in the undisturbed fluid (i.e., at infinite distance from the wave), hence we can set

$C \equiv 0$. In addition, (48.5) and (48.1) imply

$$(\partial \rho_1/\partial t) + \rho_0 \nabla^2 \phi_1 = 0. \tag{48.12}$$

Combining (48.11) and (48.12) we obtain

$$(\partial^2 \phi_1/\partial t^2) - a^2 \nabla^2 \phi_1 = 0. \tag{48.13}$$

Equations (48.7), (48.8), and (48.13) are all *wave equations*, and show that acoustic disturbances propagate as waves.

SOLUTION OF THE WAVE EQUATION

Consider the special case in which all perturbations are functions of one coordinate only (z). The wave equation then reduces to

$$(\partial^2 \phi_1/\partial t^2) - a^2(\partial^2 \phi_1/\partial z^2) = 0 \tag{48.14}$$

which has the general solution

$$\phi_1 = f_1(z - at) + f_2(z + at), \tag{48.15}$$

where f_1 and f_2 are arbitrary functions of their arguments. This solution shows that an initial disturbance $f_1(t=0) = f_{10}(z)$ propagates with unaltered shape along the positive z axis, while an initial disturbance $f_2(t=0) = f_{20}(z)$ propagates along the negative z axis, both with speed a. Thus (48.15) represents a superposition of two *traveling plane waves*, and a as defined by (48.9) is the *adiabatic speed of sound*. Moreover the only nonzero component of the wave velocity $\mathbf{v}_1 = \nabla \phi_1$ lies along the propagation axis, hence acoustic waves are *longitudinal* waves.

More generally, in Cartesian coordinates (48.13) admits solutions of the form

$$\phi_1 = f_1(\mathbf{x} \cdot \mathbf{n} - at) + f_2(\mathbf{x} \cdot \mathbf{n} + at), \tag{48.16}$$

which are plane waves traveling with speed a along $\pm\mathbf{n}$, the unit vector defining the direction of wave propagation. As implied by (48.16), the propagation of acoustic waves is isotropic because the ambient medium is homogeneous and isotropic. From (48.16) one finds

$$\mathbf{v}_1 = (\partial f_1/\partial \xi)\mathbf{n} + (\partial f_2/\partial \eta)\mathbf{n}, \tag{48.17}$$

where ξ and η denote the arguments $\mathbf{x} \cdot \mathbf{n} \mp at$; again, the waves are longitudinal, with nonzero velocity components only along $\mathbf{n}$.

For plane waves the perturbation amplitudes ρ_1, p_1, and v_1 can all be related simply. Thus choosing $\phi_1 = f(z - at)$ we have $v_1 = (\partial \phi_1/\partial z) = f'$, while (48.11) implies that $\rho_1 = -(\rho_0/a^2)(\partial \phi_1/\partial t) = (\rho_0/a)f'$. Therefore

$$\rho_1 = (v_1/a)\rho_0, \tag{48.18}$$

hence from (48.3) and (48.9)

$$p_1 = a^2 \rho_1 = a\rho_0 v_1. \tag{48.19}$$

Furthermore, using the general thermodynamic relation

$$(\partial T/\partial p)_s = \beta T/\rho c_p, \tag{48.20}$$

which follows from (2.27) and (5.7), we have

$$T_1/T_0 = (\beta/\rho_0 c_p)p_1 = (a\beta/c_p)v_1. \tag{48.21}$$

Here β is the coefficient of thermal expansion as defined by (2.14). Equations (48.18) to (48.21) show that in acoustic waves ρ_1, p_1, T_1, and v_1 are all in phase.

THE SPEED OF SOUND

Let us now derive explicit formulae for the speed of sound. For an adiabatic perfect gas, $p = p_0(\rho/\rho_0)^\gamma$ where γ is constant, hence

$$a^2 = \gamma p_0/\rho_0 = \gamma RT, \tag{48.22}$$

which shows that the speed of sound in a perfect gas is a function of the temperature only. To account for ionization effects, we merely replace γ with $\Gamma_1 \equiv (\partial \ln p/\partial \ln \rho)_s$ as given by (14.29), obtaining

$$a^2 = \Gamma_1 p_0/\rho_0. \tag{48.23}$$

For a perfect gas, (48.19) and (48.21) reduce to

$$p_1/p_0 = \gamma\rho_1/\rho_0 = \gamma v_1/a \tag{48.24a}$$

and

$$T_1/T_0 = (\gamma - 1)v_1/a. \tag{48.24b}$$

To account for ionization effects we replace γ in (48.24a) with Γ_1 and $(\gamma-1)$ in (48.24b) with $\Gamma_3 - 1 \equiv (\partial \ln T/\partial \ln \rho)_s$ as given by (14.30).

Equation (48.23) yields the correct sound speed only for a nonrelativistic fluid; hence it fails at very high temperatures, in degenerate material, or if the fluid comprises both matter and radiation, and the latter contributes significantly to the pressure and energy density of the composite fluid. To obtain a relativistically correct expression for the sound speed we linearize the relativistic dynamical equations (42.3) and (42.4), obtaining

$$(\partial \hat{e}_1/\partial t) + (\hat{e} + p)_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{48.25}$$

and

$$(\hat{e} + p)_0(\partial \mathbf{v}_1/\partial t) + c^2 \nabla p_1 = 0 \tag{48.26}$$

where $\hat{e} \equiv \rho_{00}c^2 = \rho_0(c^2 + e)$ is the total energy density of the fluid (including rest energy). Combining (48.25) and (48.26) we obtain

$$(\partial^2 \hat{e}_1/\partial t^2) - c^2 \nabla^2 p_1 = 0, \tag{48.27}$$

which implies that an acoustic wave propagates with a speed

$$a = c[(\partial p/\partial \hat{e})_s]^{1/2} = c[(\partial p/\partial \rho_0)_s(\partial \rho_0/\partial \hat{e})_s]^{1/2}. \tag{48.28}$$

But

$$(\partial\hat{e}/\partial\rho_0)_s = (\hat{e}/\rho_0) + \rho_0(\partial e/\partial\rho_0)_s, \tag{48.29}$$

and $T\,ds \equiv 0 = de - (p/\rho_0^2)\,d\rho_0$ implies that

$$(\partial e/\partial\rho_0)_s = p/\rho_0^2, \tag{48.30}$$

whence

$$(\partial\hat{e}/\partial\rho_0)_s = (\hat{e} + p)/\rho_0. \tag{48.31}$$

Therefore (48.28) becomes

$$a = c[\Gamma_1 p/(\hat{e} + p)]^{1/2}. \tag{48.32}$$

See also (**L7**), (**T4**), and (**W2**).

For a nonrelativistic gas, $\hat{e} \to \rho_0 c^2 \gg p$, and (48.32) gives $a(\text{N.R.}) = (\Gamma_1 p/\rho)^{1/2}$, in agreement with (48.23). For an extremely relativistic gas we recall from §43 that $\hat{e} \to \rho_0 e = 3p$ [cf. (43.53)] and that $\Gamma_1 \to \frac{4}{3}$, hence (48.32) gives $a(\text{E.R.}) = c/\sqrt{3}$. As we will see in §69, this result also holds for a gas composed of pure thermal radiation.

49. Propagation of Acoustic Waves

MONOCHROMATIC PLANE WAVES

Let us now consider the propagation of a *monochromatic wave* (i.e., a wave having a sinusoidal time variation at a definite frequency ω). This special case is important because an arbitrary wave packet can be synthesized from a linear combination of monochromatic waves (*Fourier components*) whose relative amplitudes and phases are determined by a Fourier analysis of the packet.

Thus consider a wave of the form

$$p_1 = P \exp[i(\omega t - \mathbf{k} \cdot \mathbf{x})], \tag{49.1}$$

$$\rho_1 = R \exp[i(\omega t - \mathbf{k} \cdot \mathbf{x})], \tag{49.2}$$

$$T_1 = \Theta \exp[i(\omega t - \mathbf{k} \cdot \mathbf{x})], \tag{49.3}$$

and

$$\phi_1 = \Phi \exp[i(\omega t - \mathbf{k} \cdot \mathbf{x})], \tag{49.4}$$

where P, R, Θ, and Φ are complex constants that are interrelated by (48.18) to (48.24). The use of complex quantities is convenient mathematically because it facilitates computation of the relative phases of different variables, but for the purposes of physical interpretation we use the *real parts* of (49.1) to (49.4).

According to (49.1) to (49.4), at a given instant the wave comprises a regular (sinusoidal) spatial sequence of compressions and rarefactions coupled to a sinusoidal velocity field. Likewise, at a given spatial point the fluid density, pressure, and temperature fluctuate sinusoidally in time around their equilibrium values, and the fluid particles oscillate sinusoidally around their equilibrium positions.

In (49.1) to (49.4) **k** is the *wave vector* or *propagation vector*, which determines the direction **n** of wave propagation via

$$\mathbf{k} = k\mathbf{n}. \tag{49.5}$$

The *wavenumber* k is related to the *wavelength* Λ of the wave by

$$k = 2\pi/\Lambda. \tag{49.6}$$

Substituting into (48.7), (48.8), and (48.13) we find that (49.1) to (49.4) are valid solutions of the wave equation provided that

$$\omega^2 = a^2k^2. \tag{49.7}$$

From (49.7) it follows that the planes of constant phase perpendicular to **n** propagate along **n** with speed a; thus for pure acoustic waves the *phase speed* v_p equals the speed of sound. Simple geometric considerations show that two planes of constant phase separated by one wavelength Λ along **n** are separated by a distance

$$\Lambda/n_i = 2\pi/kn_i = 2\pi/k_i \tag{49.8}$$

along the ith coordinate axis. Because the constant-phase surfaces succeed one another in a *period*

$$\tau = 2\pi/\omega, \tag{49.9}$$

one sees from (49.8) and (49.9) that the *phase trace speed* along the ith axis is

$$(v_t)_i = \omega/k_i = a/n_i. \tag{49.10}$$

Notice that the trace speed is infinite in the planes perpendicular to **n**, which is expected because these are planes of constant phase, hence a local change in phase at any point in a plane must "propagate" instantaneously over the entire plane (i.e., over the entire wave front). Infinite phase or trace speeds are not in violation of relativistic causality because they represent only the behavior of a mathematical "marker", not a physically significant quantity like momentum or energy.

Equations (49.5) and (49.7) show that pure acoustic waves propagate isotropically with a unique speed a, and have a simple proportionality between wavenumber and frequency. In §§52–54 we will see that the behavior of acoustic-gravity waves in a stratified medium is markedly different.

MONOCHROMATIC SPHERICAL WAVES

Thus far we have discussed only plane waves, but from symmetry considerations one expects that an isotropic medium will also support one-dimensional *spherical waves* emanating from a point source. Thus specialize (48.8) to

$$\left(\frac{\partial^2 p_1}{\partial t^2}\right)-\frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p_1}{\partial r}\right)=0. \tag{49.11}$$

The substitution $p_1 = f/r$ reduces (49.11) to $(\partial^2 f/\partial t^2) - a^2(\partial^2 f/\partial r^2)$, which leads to a general solution of the form

$$p_1 = [f_1(r-at)/r] + [f_2(r+at)/r] \tag{49.12}$$

where f_1 and f_2 are arbitrary functions. The two terms in (49.12) represent spherical disturbances diverging from, and converging on, the origin. The wave amplitude falls off as r^{-1}, hence the wave intensity (proportional to the square of the amplitude) varies as r^{-2}.

Specializing now to monochromatic waves we choose a solution of the form

$$p_1 = Pe^{i(\omega t - kr)}/r, \tag{49.13}$$

which satisfies (49.11) only if $\omega = ak$. The thermodynamic relations (48.3) and (48.19) are independent of geometry, and show that in a spherical acoustic wave both ρ_1 and T_1 are proportional to, and in phase with, p_1. From (48.11) we obtain

$$\phi_1 = (i/\rho_0\omega)p_1, \tag{49.14}$$

which shows that ϕ_1 leads p_1 in phase by 90°. Calculating $\mathbf{v}_1$ from (48.5) we have

$$\mathbf{v}_1 = [k-(i/r)](p_1/\rho_0\omega)\hat{\mathbf{r}}. \tag{49.15}$$

As $r \to \infty$, (49.15) reduces to (48.19), as it should because a wave is locally planar when its radius of curvature approaches infinity. However, near the origin v_1 lags p_1 by 90°, and grows in amplitude as r^{-2}.

GROUP VELOCITY

To determine the speed of energy propagation in waves we study the behavior of *wave packets*. A packet is a localized disturbance that can be described mathematically as a superposition of a large (perhaps infinite) number of Fourier components that interfere in such a way as to produce a finite wave amplitude only in a strongly localized region in space and time. Intuitively one expects that such a localized disturbance must result from a concentration of wave energy, and that the motion of the packet tracks the flow of wave energy in the medium.

Consider a packet composed of plane waves $\exp[i(\omega t - \mathbf{k}\cdot\mathbf{x})]$ whose wave vectors $\mathbf{k}$ lie in the volume $\mathbf{k}_0 \pm \Delta\mathbf{k}$ of $\mathbf{k}$ space; we assume that all waves having their wave vectors within this volume have unit amplitude, and that all others have zero amplitude. We further assume that ω is a general function of (k_x, k_y, k_z). The amplitude of the wave packet is then

$$A(\mathbf{x}, t) = \int_{k_{0x}-\Delta k_x}^{k_{0x}+\Delta k_x} dk_x \int_{k_{0y}-\Delta k_y}^{k_{0y}+\Delta k_y} dk_y \int_{k_{0z}-\Delta k_z}^{k_{0z}+\Delta k_z} dk_z e^{i[\omega(k_x,k_y,k_z)t - k_x x - k_y y - k_z z]}. \tag{49.16}$$

For a small enough volume $\Delta\mathbf{k}$, we can use the linear expansion

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + (\partial\omega/\partial k_x)_0\, \delta k_x + (\partial\omega/\partial k_y)_0\, \delta k_y + (\partial\omega/\partial k_z)_0\, \delta k_z, \tag{49.17}$$

whence

$$\begin{aligned} A(\mathbf{x}, t) &\approx e^{i(\omega_0 t - \mathbf{k}_0\cdot\mathbf{x})} \int_{-\Delta k_x}^{\Delta k_x} d(\delta k_x) \int_{-\Delta k_y}^{\Delta k_y} d(\delta k_y) \int_{-\Delta k_z}^{\Delta k_z} d(\delta k_z) \\ &\times \exp\left[\left[i\left\{\left[\left(\frac{\partial\omega}{\partial k_x}\right)_0 t - x\right]\delta k_x + \left[\left(\frac{\partial\omega}{\partial k_y}\right)_0 t - y\right]\delta k_y + \left[\left(\frac{\partial\omega}{\partial k_z}\right)_0 t - z\right]\delta k_z\right\}\right]\right] \\ &\approx 8e^{i(\omega_0 t - \mathbf{k}_0\cdot\mathbf{x})} \frac{\sin\{[(\partial\omega/\partial k_x)_0 t - x]\,\Delta k_x\}}{(\partial\omega/\partial k_x)_0 t - x} \\ &\times \frac{\sin\{[(\partial\omega/\partial k_y)_0 t - y]\,\Delta k_y\}}{(\partial\omega/\partial k_y)_0 t - y} \\ &\times \frac{\sin\{[(\partial\omega/\partial k_z)_0 t - z]\,\Delta k_z\}}{(\partial\omega/\partial k_z)_0 t - z}. \end{aligned} \tag{49.18}$$

Equation (49.18) shows that the packet is an *amplitude-modulated plane wave.* The maximum of each of the factors $(\sin\xi)/\xi$ is attained as $\xi \to 0$, hence the packet has maximum amplitude at

$$x = (\partial\omega/\partial k_x)_0 t, \tag{49.19a}$$

$$y = (\partial\omega/\partial k_y)_0 t, \tag{49.19b}$$

and

$$z = (\partial\omega/\partial k_z)_0 t. \tag{49.19c}$$

Thus the wave packet has a *group velocity* (or *packet velocity*)

$$\mathbf{v}_g = \boldsymbol{\nabla}_k \omega \tag{49.20}$$

where

$$\boldsymbol{\nabla}_k \equiv (\partial/\partial k_x)\hat{\mathbf{i}} + (\partial/\partial k_y)\hat{\mathbf{j}} + (\partial/\partial k_z)\hat{\mathbf{k}}. \tag{49.21}$$

These results hold for an arbitrary dependence of ω on $\mathbf{k}$.

In the particular case of a packet of pure acoustic waves, for which $\omega = ak$, one readily finds

$$\mathbf{v}_g = a[(k_x/k)_0\hat{\mathbf{i}} + (k_y/k)_0\hat{\mathbf{j}} + (k_z/k)_0\hat{\mathbf{k}}] = a\mathbf{n}_0. \tag{49.22}$$

Thus, for pure acoustic waves, the group velocity equals the phase velocity, and an acoustic wave packet propagates with the speed of sound along $\mathbf{n}_0$ (or $\mathbf{k}_0$); as we will see in §53, the behavior of gravity-modified acoustic waves in a stratified medium is more complicated (and interesting).

50. Wave Energy and Momentum

We obtained the mechanical energy equation for a fluid by taking the dot product of the flow velocity with the momentum equation (cf. §§24 and 27). Similarly we can derive a mechanical energy equation for an acoustic wave by taking the dot product of the velocity fluctuation $\mathbf{v}_1$ with the linearized momentum equation (48.2), obtaining

$$\rho_0 \mathbf{v}_1 \cdot (\partial \mathbf{v}_1/\partial t) = (\tfrac{1}{2}\rho_0 v_1^2)_{,t} = -\mathbf{v}_1 \cdot \nabla p_1, \tag{50.1}$$

which states that the rate of change of the kinetic energy density in the wave equals the rate of work done on the fluid by the wave-induced pressure gradient. Notice that all quantities in (50.1) are *second order.*

In place of the gas energy equation, we have the adiabatic relation (48.3), which can be combined with the continuity equation (48.1), to give

$$(p_1/a^2\rho_0)(\partial p_1/\partial t) = (\tfrac{1}{2}p_1^2/a^2\rho_0)_{,t} = -p_1 \nabla \cdot \mathbf{v}_1. \tag{50.2}$$

In order to interpret (50.2) physically, consider the energy density $\hat{e} = \rho e$ of the disturbed fluid. If we take $e = e(\rho, s)$, then because the wave is adiabatic

$$e\rho = e_0\rho_0 + [\partial(\rho e)/\partial \rho]_s \rho_1 + \tfrac{1}{2}[\partial^2(\rho e)/\partial \rho^2]_s \rho_1^2. \tag{50.3}$$

But from (3.12), $(\partial e/\partial \rho)_s = p/\rho^2$, hence $[\partial(\rho e)/\partial \rho]_s = h$. Therefore

$$[\partial^2(\rho e)/\partial \rho^2]_s = (\partial h/\partial \rho)_s = (\partial h/\partial p)_s(\partial p/\partial \rho)_s = a^2/\rho, \tag{50.4}$$

where we used (2.33) to obtain $(\partial h/\partial p)_s$. Thus

$$\rho e = \rho_0 e_0 + h_0\rho_1 + \tfrac{1}{2}(a^2\rho_1^2/\rho_0) = \rho_0 e_0 + h_0 \rho_1 + \tfrac{1}{2}(p_1^2/a^2\rho_0). \tag{50.5}$$

The first term on the right-hand side of (50.5) is the energy density of the unperturbed fluid and is unrelated to the presence of the wave. The second term averages to zero over a sufficiently large volume because the total mass of the fluid cannot be changed by a wave, hence $\int \rho_1\, dV = 0$. (Likewise, this term averages to zero over time for harmonic disturbances.) Hence the third term represents the nonvanishing net change in the fluid's internal energy density resulting from the presence of a wave; this energy (a second-order quantity) is called the *compressional energy* of the wave. Thus (50.2) is analogous to the first law of thermodynamics, stating that the rate of change of the compressional energy stored in the wave equals the negative of the rate of work done by the wave's pressure perturbation on the wave-induced expansion and compression of the fluid.

Taking the sum of (50.1) and (50.2) we obtain a *wave energy equation* in conservation form, namely

$$(\partial \varepsilon_w/\partial t) = -\nabla \cdot \boldsymbol{\phi}_w \tag{50.6}$$

where the *wave energy density* is

$$\varepsilon_w = \tfrac{1}{2}\rho_0 v_1^2 + \tfrac{1}{2}(p_1^2/a^2\rho_0) \tag{50.7}$$

and the *wave energy flux* is

$$\boldsymbol{\phi}_w = p_1\mathbf{v}_1. \tag{50.8}$$

The momentum density in the perturbed fluid is $\boldsymbol{\mu} = \rho\mathbf{v} = \rho_0\mathbf{v}_1 + \rho_1\mathbf{v}_1$. Using the fact that $\mathbf{v}_1 = \nabla\phi_1$, one sees from equation (A2.64) that

$$\rho_0 \int_V \mathbf{v}_1 \, dV = \rho_0 \int_S \phi_1 \mathbf{n} \, dS. \tag{50.9}$$

The latter integral is identically zero if S lies in the unperturbed fluid where $\phi_1 \equiv 0$, so in a sufficiently large volume the net *wave momentum density* is

$$\boldsymbol{\mu}_w = \rho_1\mathbf{v}_1 = p_1\mathbf{v}_1/a^2 = \boldsymbol{\phi}_w/a^2. \tag{50.10}$$

The above formulae simplify for plane waves. Thus from (48.19) we find

$$\varepsilon_w = \rho_0 v_1^2 \tag{50.11}$$

and

$$\boldsymbol{\phi}_w = a\rho_0 v_1^2 \mathbf{n} = a\varepsilon_w \mathbf{n}. \tag{50.12}$$

The *instantaneous* values of ε_w and ϕ_w are of little interest; rather we usually wish to know the *time averages* $\langle\varepsilon_w\rangle$ and $\langle\phi_w\rangle$. In particular, for monochromatic waves, we calculate averages over a cycle (one wave period). When complex representations like (49.1) to (49.4) are used it is important to remember that in computing, say, $\langle p_1\mathbf{v}_1\rangle$ we must take the time average of the *real parts* of p_1 and $\mathbf{v}_1$. This is most easily done by noting that for any time-harmonic complex quantities $\alpha = \alpha_0 e^{i\omega t} = \alpha_R + i\alpha_I$ and $\beta = \beta_0 e^{i(\omega t + \psi)} = \beta_R + i\beta_I$ we have the general identities

$$\tfrac{1}{2}(\alpha^*\beta + \alpha\beta^*) \equiv \alpha_0\beta_0 \cos\psi \tag{50.13}$$

and

$$\begin{aligned} \langle\alpha_R\beta_R\rangle &= \alpha_0\beta_0\langle\cos(\omega t)\cos(\omega t + \psi)\rangle \equiv \tfrac{1}{2}\alpha_0\beta_0\cos\psi \\ &\equiv \alpha_0\beta_0\langle\sin(\omega t)\sin(\omega t + \psi)\rangle = \langle\alpha_I\beta_I\rangle. \end{aligned} \tag{50.14}$$

Hence for monochromatic plane waves we have the useful result

$$\langle\alpha_R\beta_R\rangle \equiv \tfrac{1}{4}(\alpha^*\beta + \alpha\beta^*). \tag{50.15}$$

From (50.15) we see that for a monochromatic wave the average kinetic and potential energy densities each equal

$$\tfrac{1}{2}\rho_0\langle v_{1R}^2\rangle = \tfrac{1}{4}\rho_0 v_1 v_1^* = \tfrac{1}{4}\rho_0 \mathbf{v}_1 \cdot \mathbf{v}_1^*, \tag{50.16}$$

hence the average wave energy density is

$$\langle \varepsilon_w \rangle = \tfrac{1}{2}\rho_0 \mathbf{v}_1 \cdot \mathbf{v}_1^*, \tag{50.17}$$

and the average wave energy flux is

$$\langle \boldsymbol{\phi}_w \rangle = \langle p_{1R} \mathbf{v}_{1R} \rangle = \tfrac{1}{4}(p_1 \mathbf{v}_1^* + p_1^* \mathbf{v}_1). \tag{50.18}$$

For a wave packet whose velocity amplitude can be represented by a sum over monochromatic components, that is,

$$\mathbf{V}(t) = \sum_j \mathbf{V}_j e^{i\omega_j t}, \tag{50.19}$$

one easily sees that the only terms surviving in a time average yield

$$\langle V_R^2 \rangle = \tfrac{1}{2}\langle \mathbf{V} \cdot \mathbf{V}^* \rangle = \tfrac{1}{2}\sum_j \mathbf{V}_j \cdot \mathbf{V}_j^*. \tag{50.20}$$

Hence the average energy density in the wave equals the sum of the average energy densities in the monochromatic components. Likewise

$$\langle \boldsymbol{\phi}_w \rangle = \sum_j \langle \boldsymbol{\phi}_{wj} \rangle. \tag{50.21}$$

51. Damping of Acoustic Waves by Conduction and Viscosity

Thus far we have ignored viscosity and thermal conduction and have supposed that acoustic waves propagate adiabatically. We now inquire what happens to a wave when these dissipative processes are operative. We assume the fluid is a perfect gas, and consider the propagation of a plane wave along the x axis.

The linearized continuity equation (48.1) is unaffected by viscosity or conductivity, while the linearized momentum equation (26.2) is

$$\rho_0(\partial v_1/\partial t) = -(\partial p_1/\partial x) + \mu'(\partial^2 v_1/\partial x^2) \tag{51.1}$$

and the linearized energy equation (27.11) is

$$\rho_0(\partial e_1/\partial t) = -p_0(\partial v_1/\partial x) + K(\partial^2 T_1/\partial x^2). \tag{51.2}$$

Here $\mu' \equiv \frac{4}{3}\mu + \zeta$ is the effective one-dimensional viscosity. Notice that no viscous term appears in (51.2) because the dissipation function $\mu'(\partial v_1/\partial x)^2$ is a second-order quantity.

For a perfect gas $p = \rho RT = (\gamma - 1)\rho e$, hence

$$T_1 = [p_1 - (p_0/\rho_0)\rho_1]/R\rho_0 \tag{51.3}$$

and

$$e_1 = [p_1 - (p_0/\rho_0)\rho_1]/(\gamma - 1)\rho_0. \tag{51.4}$$

Using (51.3) and (51.4) and eliminating $(\partial v_1/\partial x)$ via (48.1), we can rewrite (51.2) as

$$\frac{\partial}{\partial t}(p_1 - a^2\rho_1) = \gamma\chi \frac{\partial^2}{\partial x^2}(p_1 - a^2\rho_1), \tag{51.5}$$

where χ is the *thermal diffusivity* [cf. (28.3)]

$$\chi \equiv K/\rho_0 c_p = (\gamma - 1)K/\gamma R\rho_0. \tag{51.6}$$

Now take a plane-wave solution

$$p_1 = P \exp[i(\omega t - kx)] \tag{51.7}$$

$$\rho_1 = R \exp[i(\omega t - kx)] \tag{51.8}$$

and

$$\phi_1 = \Phi \exp[i(\omega t - kx)], \tag{51.9}$$

where $v_1 = (\partial\phi_1/\partial x)$. Substituting (51.7) to (51.9) in (48.1), (51.1), and (51.5) we find

$$i\omega R - \rho_0 k^2 \Phi = 0, \tag{51.10}$$

$$-ikP + (\rho_0 k\omega - i\mu' k^3)\Phi = 0, \tag{51.11}$$

and

$$(i\omega + \gamma\chi k^2)P - a^2(i\omega + \chi k^2)R = 0. \tag{51.12}$$

We obtain a nontrivial solution of (51.10) to (51.12) only if the determinant of the coefficients vanishes. Enforcing this condition, we obtain the *dispersion relation*

$$\omega^2 = a^2 k^2 \left[\frac{1 - i(\chi k^2/\omega)}{1 - i(\gamma\chi k^2/\omega)}\right] + \frac{i\mu' k^2 \omega}{\rho_0}. \tag{51.13}$$

Notice that when $\chi = \mu' = 0$ we recover our previous result $\omega^2 = a^2 k^2$.

Suppose first that both μ' and χ are very small. Then set $k = k_0 + \delta k = (\omega/a) + \delta k$, expand to first order in small quantities, and solve for δk to obtain

$$\delta k = -(i\omega^2/2a^3\rho_0)[\mu' + (\gamma - 1)\rho_0\chi]. \tag{51.14}$$

The solution for, say, p_1 is then of the form

$$p_1 = P e^{i(\omega t - k_0 x)} e^{-x/L} \tag{51.15}$$

where $L \equiv -i/\delta k$, and similarly for ρ_1, T_1, ϕ_1, and v_1.

Equation (51.15) shows that the wave still propagates with the sound speed a, but its amplitude steadily diminishes with a characteristic decay-length L. Thus small amounts of viscosity and conductivity *damp* acoustic waves by an irreversible conversion of wave energy into entropy. Recalling from §29 that $(\mu/\rho) \sim \chi \sim a\lambda$, where λ is a particle mean free path, we see that $(L/\Lambda) \sim (\Lambda/\lambda)$, hence the decay length is of the order of (Λ/λ) wavelengths. The assumptions under which we solved (51.13) imply $(\Lambda/\lambda) \gg 1$, hence the decay is slow. Equation (51.14) shows that high-frequency waves are more heavily damped than low-frequency waves, which is not surprising because for a given amplitude they will have steeper gradients of velocity and temperature over smaller physical distances (a wavelength).

In reality the physics of the problem is more complicated, and more interesting, than indicated above. To simplify the analysis suppose the gas has zero viscosity but a finite thermal conductivity. (Recalling from §§29 and 33 that in a real gas $K \propto \mu$ this assumption may, at first sight, seem hypothetical. But as we will see in §101 it is actually realistic for a radiating gas where radiation provides an efficient energy transport mechanism while viscosity and thermal conduction—by particles—are both negligible). The dispersion relation then becomes

$$k^4 - [(\gamma\omega^2/a^2) - i(\omega/\chi)]k^2 - i(\omega^3/a^2\chi) = 0, \tag{51.16}$$

which yields immediately

$$2k^2 = \left(\frac{\gamma\omega^2}{a^2} - \frac{i\omega}{\chi}\right) \pm \left[\frac{\gamma^2\omega^4}{a^4} - \frac{\omega^2}{\chi^2} - \frac{2i(\gamma-2)\omega^3}{a^2\chi}\right]^{1/2}. \tag{51.17}$$

In general, one must calculate $k(\omega)$ from (51.17) numerically. But we can obtain analytical expressions in two limiting regimes. First, suppose that the dimensionless ratio $\varepsilon \equiv (\omega\chi/a^2) \ll 1$ because ω and/or χ is very small; from the scaling relation $\chi \sim a\lambda$ one recognizes that in this regime $\lambda/\Lambda \ll 1$. Expanding (51.17) to second order in ε we have

$$2k^2 \approx (\omega/\chi)(\gamma\varepsilon - i) \pm (i\omega/\chi)[1 + i(\gamma-2)\varepsilon - 2(\gamma-1)\varepsilon^2]. \tag{51.18}$$

Taking the positive root we have

$$k_+^2 \approx (\omega/a)^2[1 - i(\gamma-1)(\chi\omega/a^2)] \tag{51.19}$$

whence we obtain

$$k_1 \approx \pm[(\omega/a) - i(\gamma-1)(\chi\omega^2/2a^3)]. \tag{51.20}$$

This root corresponds to the same mode obtained in (51.14), that is, a slowly damped acoustic wave propagating with the adiabatic sound speed.

Taking the negative root in (51.18) and retaining only leading terms we find

$$k_-^2 \approx -i\omega/\chi \tag{51.21}$$

whence we obtain

$$k_2 \approx \pm(\omega/2\chi)^{1/2}(1-i). \tag{51.22}$$

This root corresponds to a new mode, a propagating *thermal wave* [cf. (**L2**, §77) and §§101 and 103]. This particular mode propagates very slowly, with phase speed $v_p = (2\varepsilon)^{1/2}\, a \ll a$; because the real and imaginary parts of k are of the same magnitude, the wave is very heavily damped, decaying away in a few wavelengths. Moreover $\Lambda_2/\Lambda_0 = k_0/k_{2R} = (2\varepsilon)^{1/2} \ll 1$, hence the physical distance over which this mode can propagate is less than a wavelength of an undamped acoustic wave of the same frequency.

Suppose now that the dimensionless ratio $\varepsilon' \equiv (a^2/\omega\chi) \ll 1$ because ω and/or χ is very large; estimating χ as before we see that in this regime

$\Lambda/\lambda \ll 1$. We now expand (51.17) as

$$2k^2 \approx \left(\frac{\gamma\omega^2}{a^2}\right)\left(1-\frac{i\varepsilon'}{\gamma}\right) \pm \left(\frac{\gamma\omega^2}{a^2}\right)\left[1+\frac{i(2-\gamma)\varepsilon'}{\gamma^2}\right]. \tag{51.23}$$

Taking the positive root we have

$$k_+^2 \approx (\omega^2/a_T^2)[1-i(\gamma-1)(\varepsilon'/\gamma^2)], \tag{51.24}$$

whence

$$k_3 \approx \pm[(\omega/a_T) - i(\gamma-1)(a_T/2\gamma\chi)]. \tag{51.25}$$

Here a_T is the *isothermal sound speed*

$$a_T^2 \equiv (\partial p/\partial \rho)_T = p/\rho = a^2/\gamma. \tag{51.26}$$

This root again corresponds to a damped acoustic wave, but with the qualitatively important difference that now the wave propagates at *constant temperature* because wave-induced temperature fluctuations are efficiently

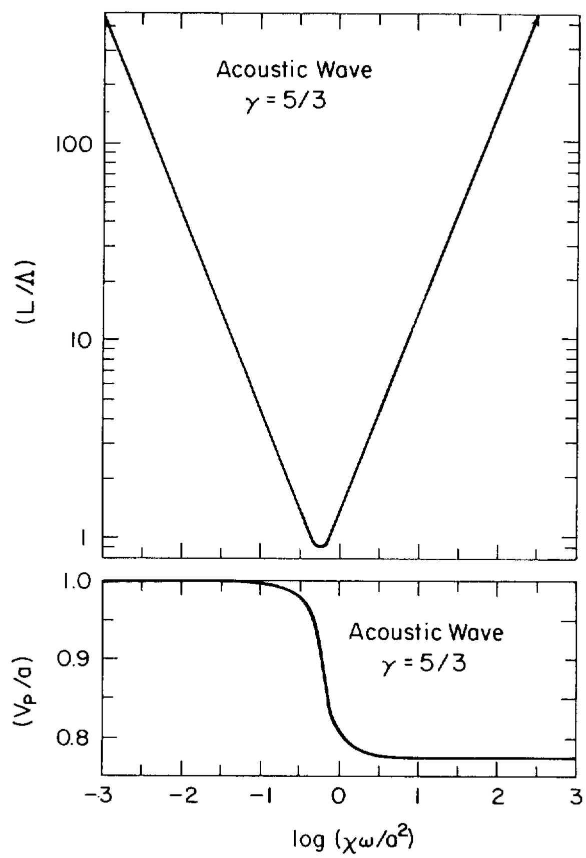

Fig. 51.1 Damping length and phase velocity of damped acoustic mode.

eradicated by the high thermal conductivity and/or steep temperature gradients. Isothermal acoustic waves propagate a factor of $\gamma^{-1/2}$ more slowly than adiabatic acoustic waves; the decay length is now independent of frequency and is proportional to the thermal conductivity. Because $|k_{3I}/k_{3R}| \ll 1$ the wave survives over many wavelengths before it decays. This is not to say that it is slowly damped, however, because by scaling arguments one finds that $L \approx \lambda$, hence the wave decays significantly over a particle mean free path!

Taking the negative root in (51.23) we have

$$k_-^2 \approx -i\omega/\gamma\chi, \tag{51.27}$$

whence we obtain

$$k_4 \approx \pm(\omega/2\gamma\chi)^{1/2}(1-i). \tag{51.28}$$

This root corresponds to another thermal wave, which is heavily damped in the sense that it decays away over a few wavelengths. However the phase

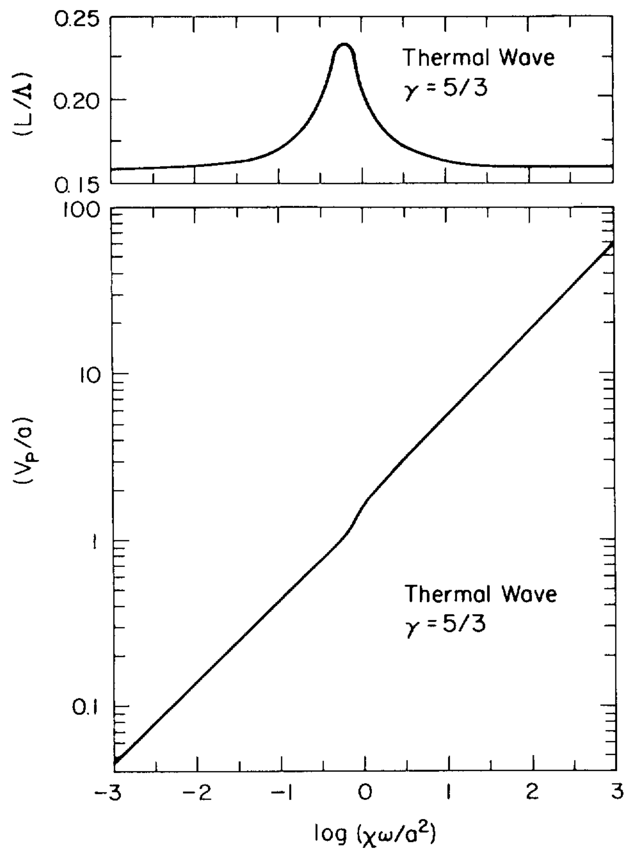

Fig. 51.2 Damping length and phase velocity of thermal mode.

speed of this mode is very large, $v_p=(2\gamma\chi\omega)^{1/2}=(2\gamma/\varepsilon')^{1/2}\ a\gg a$, as is its wavelength compared to that of an undamped acoustic wave of the same frequency: $\Lambda_4/\Lambda_0=(2\gamma/\varepsilon')^{1/2}\gg 1$. Hence this mode propagates rapidly over a physical distance of the order of $(a\lambda/\omega)^{1/2}$ before it is damped.

Plots of v_p/a and L/Λ obtained from a numerical solution of (51.17) for $\gamma=\frac{5}{3}$ are shown in Figures 51.1 and 51.2. Here one sees that the acoustic mode is most heavily damped when $(\chi\omega/a^2)\sim 1$, which is also where v_p drops from a to a_T for that mode. In contrast, the thermal wave mode is least damped when $(\chi\omega/a^2)\sim 1$.

The damping of acoustic waves in relativistic fluids is discussed by Weinberg (**W2**), who gives an expression for δk [cf. his equations (2.55) to (2.57)] which reduces to (51.14) in the nonrelativistic limit.

5.2 Acoustic-Gravity Waves

Let us now consider wave propagation in a compressible medium stably stratified in a gravitational field, such as the atmosphere of the Earth, the Sun, or a star. In such a medium, waves can be driven by both compressional and *buoyancy* restoring forces, each of which can induce harmonic oscillations of a fluid element slightly displaced from its equilibrium position. As a result the behavior of waves is more complicated than in a homogeneous medium: (1) Their propagation characteristics are *anisotropic* because the force of gravity imposes a preferred direction in the fluid. (2) They are *dispersive* (i.e., the propagation speed varies as k and/or ω are varied). (3) Stratification of the atmosphere imposes a *cutoff frequency* below which *gravity-modified acoustic waves* cannot propagate, and thus restricts the region of $(k_x-\omega)$ space in which such waves can exist. (4) Buoyancy forces give rise to a new class of propagating waves: the low-frequency pressure-modified *internal gravity waves,* which are inherently *two-dimensional.* (The adjective "internal" is used to discriminate these waves from surface gravity waves found at fluid interfaces, for example, the surface of the ocean; for brevity we refer to these two different wave modes simply as "acoustic waves" and "gravity waves".)

Acoustic-gravity waves are of interest because they are ubiquitous in the terrestrial and solar atmospheres, and certainly must exist in stellar atmospheres as well. In §§52 to 54 we consider adiabatic fluctuations because the effects of viscosity and thermal conduction on acoustic-gravity waves are negligible in astrophysical media. The effects of radiative damping, which can be major, are discussed in §102.

52. The Wave Equation and Wave Energy

BUOYANCY OSCILLATIONS

We can obtain insight into buoyancy effects by considering the motion of a small fluid parcel rising and falling adiabatically in a stratified medium. We assume strictly vertical motion and write the vertical displacement as ζ_1. When the parcel is displaced from its equilibrium position it experiences a net force $-g(\rho-\rho_0)$ where ρ is the density inside the parcel and ρ_0 is the density in the ambient atmosphere; hence its motion is governed by

$$\rho_1(\partial^2\zeta_1/\partial t^2)=-g(\rho-\rho_0). \tag{52.1}$$

For small displacements

$$\rho_0(\zeta_1)=\rho_0(0)+(d\rho/dz)_{at}\zeta_1 \tag{52.2a}$$

and

$$\rho(\zeta_1)=\rho_0(0)+(d\rho/dz)_{ad}\zeta_1, \tag{52.2b}$$

where the subscripts denote "atmosphere" and "adiabatic". Thus

$$(\partial^2\zeta_1/\partial t^2)=-\omega_{\rm BV}^2\zeta_1 \tag{52.3}$$

where

$$\omega_{\rm BV}^2\equiv(g/\rho)[(d\rho/dz)_{ad}-(d\rho/dz)_{at}] \tag{52.4}$$

is known as the *Brunt-Väisälä frequency.*

Equation (52.3) admits two distinct classes of solutions:

$$\zeta_1(t)=\zeta_1(0)\exp(\pm i\,|\omega_{\rm BV}|\,t) \tag{52.5a}$$

for $\omega_{\rm BV}^2>0$, and

$$\zeta_1(t)=\zeta_1(0)\exp(\pm\,|\omega_{\rm BV}|\,t) \tag{52.5b}$$

for $\omega_{\rm BV}^2<0$.

When $\omega_{\rm BV}^2>0$ the atmosphere is *stably stratified* and fluid parcels undergo harmonic oscillations of bounded amplitude. But when $\omega_{\rm BV}^2<0$, a displaced fluid parcel experiences further force in the direction of its displacement, and the perturbation grows exponentially. In the latter case the atmosphere is *convectively unstable*; indeed the inequality $(d\rho/dz)_{ad}<(d\rho/dz)_{at}$ is just the standard *Schwarzschild criterion* for instability against convection (**C9,** 264). When $\omega_{\rm BV}^2=0$ the atmosphere is in adiabatic equilibrium, and is *neutrally stable*; neither buoyancy oscillations nor convection can then occur.

In a simple buoyancy oscillation the displaced fluid expands and cools adiabatically as it rises, and slows in response to gravitational braking as the density in the parcel exceeds that in the surrounding medium. At the top of the cycle $T_1<0$, $\rho_1>0$, and $v_1=0$. The denser fluid element is then accelerated downward, and passes through its equilibrium position, where $\rho_1=T_1=0$, with maximum downward velocity. Thus ρ_1, T_1, and v_1 are out of phase, T_1 leading v_1 by 90° and ρ_1 lagging v_1 by 90°, in strong contrast with a pure acoustic wave where ρ_1, T_1, and v_1 are all exactly in phase (cf. §48).

Because buoyancy oscillations are slow (see §§53 and 54), sound waves have time to run back and forth within a displaced fluid parcel and to establish pressure equilibrium between it and the surrounding atmosphere.

We can therefore take the pressure gradient in the parcel to be the same as in the unperturbed atmosphere, and using the relation $\delta\rho = (\partial\rho/\partial p)_{ad}\,\delta p = \delta p/a^2$ inside the parcel we can rewrite (52.4) in the useful form

$$\omega_{\rm BV}^2 = (g/a^2\rho)[(dp/dz)_{at} - a^2(d\rho/dz)_{at}]. \qquad (52.6)$$

FLUID EQUATIONS

The dynamical behavior of acoustic-gravity waves is determined by the equation of continuity (19.4), Euler's equation (23.6) with $\mathbf{f} \equiv \rho\mathbf{g}$, and the gas energy equation

$$\rho\{(De/Dt) + p[D(1/\rho)/Dt]\} = (Dq/Dt), \qquad (52.7)$$

where (Dq/Dt) is the net rate of energy input, per unit volume, to the gas from external sources. For adiabatic fluctuations $(Dq/Dt) \equiv 0$.

Before writing linearized fluid equations, it is convenient to derive some alternative forms of (52.7), which will prove useful later. We first develop some necessary thermodynamic relations. Thus expanding dp as a function of (ρ, T) we have the general expression

$$(d \ln p/d \ln \rho) = (\partial \ln p/\partial \ln T)_\rho (d \ln T/d \ln \rho) + (d \ln p/d \ln \rho)_T. \qquad (52.8)$$

Using the cyclic relation (2.7) one finds $(\partial p/\partial T)_\rho = \beta/\kappa_T$ where β and κ_T are defined by (2.14) and (2.15). We can thus rewrite (52.8) as

$$(d \ln p/d \ln \rho) = (T\beta/p\kappa_T)(d \ln T/d \ln \rho) + (1/p\kappa_T). \qquad (52.9)$$

This relation is general, hence holds for adiabatic changes in particular. Thus using (14.19) and (14.21) we obtain the important identity

$$\Gamma_1 = (T\beta/p\kappa_T)(\Gamma_3 - 1) + (1/p\kappa_T). \qquad (52.10)$$

Next, we obtain a general expression for the ratio c_p/c_v by applying (5.17) to an adiabatic process, which yields

$$c_p/c_v = \rho\kappa_T(\partial p/\partial\rho)_s = p\kappa_T\Gamma_1. \qquad (52.11)$$

Substituting (52.10) and (52.11) into (5.7) we then find

$$c_v = \beta/\kappa_T\rho(\Gamma_3 - 1) \qquad (52.12)$$

whence, using (14.22) and (52.11) we obtain

$$c_p = (\beta p/\rho)\Gamma_2/(\Gamma_2 - 1). \qquad (52.13)$$

Suppose now we choose T and p as fundamental variables in (52.7). Then using (2.15), (2.22), and (5.11) in the expansion

$$\rho\left[\left(\frac{\partial e}{\partial T}\right)_p - \frac{p}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p\right] dT + \rho\left[\left(\frac{\partial e}{\partial p}\right)_T - \frac{p}{\rho^2}\left(\frac{\partial\rho}{\partial p}\right)_T\right] dp = dq \qquad (52.14)$$

we have

$$c_p\rho\, dT - \beta T\, dp = dq \tag{52.15}$$

which, with the aid of (52.13), can be rewritten as

$$c_p\rho T\left[\frac{dT}{T} - \frac{(\Gamma_2 - 1)}{\Gamma_2}\frac{dp}{p}\right] = dq. \tag{52.16}$$

Alternatively, suppose we choose p and ρ as fundamental variables. Then using (2.28), (2.29), and (52.11) in the expansion

$$\rho\left(\frac{\partial e}{\partial p}\right)_\rho dp + \rho\left[\left(\frac{\partial e}{\partial \rho}\right)_p - \frac{p}{\rho^2}\right] d\rho = dq \tag{52.17}$$

we get

$$(\kappa_T c_v \rho/\beta)\, dp - (c_p/\beta)\, d\rho = (\kappa_T c_v \rho/\beta)[dp - (\Gamma_1 p/\rho)\, d\rho] = dq. \tag{52.18}$$

Then using (52.12) and (48.23) we find

$$(\Gamma_3 - 1)^{-1}(dp - a^2\, d\rho) = dq. \tag{52.19}$$

Finally, suppose we choose ρ and T as fundamental variables. Then using (2.11), (2.17), and (5.7) in the expansion

$$\rho\left(\frac{\partial e}{\partial T}\right)_\rho dT + \rho\left[\left(\frac{\partial e}{\partial \rho}\right)_T - \frac{p}{\rho^2}\right] d\rho = dq \tag{52.20}$$

we find

$$\rho c_v\, dT - (\beta T/\kappa_T \rho)\, d\rho = dq. \tag{52.21}$$

Hence from (52.12)

$$\rho c_v T[(dT/T) - (\Gamma_3 - 1)(d\rho/\rho)] = dq. \tag{52.22}$$

LINEARIZED FLUID EQUATIONS

Assume that the ambient atmosphere is static and in hydrostatic equilibrium so that

$$\nabla p_0 = \rho_0 \mathbf{g}, \tag{52.23}$$

where $\mathbf{g} = -g\mathbf{k}$ is constant. Then for small-amplitude adiabatic disturbances the linearized fluid equations are

$$(\partial\rho_1/\partial t) + \nabla\cdot(\rho_0\mathbf{v}_1) = 0, \tag{52.24}$$

$$\rho_0(\partial\mathbf{v}_1/\partial t) = \rho_1\mathbf{g} - \nabla p_1, \tag{52.25}$$

and, from (52.19),

$$(Dp_1/Dt) - a^2(D\rho_1/Dt) = 0, \tag{52.26}$$

which can be rewritten as

$$(p_1 - a^2\rho_1)_{,t} + \mathbf{v}_1\cdot(\nabla p_0 - a^2\nabla\rho_0) = 0 \tag{52.27}$$

or

$$\begin{aligned}(\partial p_1/\partial t) &= -\mathbf{v}_1 \cdot \nabla p_0 - a^2 \rho_0 \nabla \cdot \mathbf{v}_1 \\ &= -\mathbf{v}_1 \cdot \nabla p_0 - \Gamma_1 p_0 \nabla \cdot \mathbf{v}_1. \end{aligned} \tag{52.28}$$

WAVE EQUATION

To derive a wave equation we first differentiate (52.25) with respect to time, obtaining

$$\rho_0(\partial^2 \mathbf{v}_1/\partial t^2) = (\partial \rho_1/\partial t)\mathbf{g} - \nabla(\partial p_1/\partial t). \tag{52.29}$$

We then eliminate $(\partial \rho_1/\partial t)$ and $(\partial p_1/\partial t)$ via (52.24) and (52.28); after some simple reductions one finds

$$\begin{aligned}(\partial^2 \mathbf{v}_1/\partial t^2) = {} & a^2\, \nabla(\nabla \cdot \mathbf{v}_1) + (a^2\, \nabla \cdot \mathbf{v}_1)\nabla \ln \Gamma_1 + (\Gamma_1 - 1)(\nabla \cdot \mathbf{v}_1)\mathbf{g} \\ & + \rho_0^{-1}[\nabla(\mathbf{v}_1 \cdot \nabla p_0) - \mathbf{g}(\mathbf{v}_1 \cdot \nabla \rho_0)]. \end{aligned} \tag{52.30}$$

In the ambient atmosphere there is a unique relation between p_0 and ρ_0, that is, we can write $\rho_0 = f(p_0)$. Hence $\nabla \rho_0 = (df/dp_0)\nabla p_0 = (df/dp_0)\rho_0 \mathbf{g}$; therefore (52.30) reduces to

$$(\partial^2 \mathbf{v}_1/\partial t^2) = a^2\, \nabla(\nabla \cdot \mathbf{v}_1) + (a^2\, \nabla \cdot \mathbf{v}_1)\nabla \ln \Gamma_1 + (\Gamma_1 - 1)(\nabla \cdot \mathbf{v}_1)\mathbf{g} + \nabla(\mathbf{g} \cdot \mathbf{v}_1), \tag{52.31}$$

which is the fundamental equation governing the propagation of acoustic-gravity waves. For a gas with constant ratio of specific heats, $\Gamma_1 \equiv \gamma$, and (52.31) simplifies to

$$(\partial^2 \mathbf{v}_1/\partial t^2) = a^2\, \nabla(\nabla \cdot \mathbf{v}_1) + (\gamma - 1)(\nabla \cdot \mathbf{v}_1)\mathbf{g} + \nabla \cdot (\mathbf{g} \cdot \mathbf{v}_1), \tag{52.32}$$

an expression first derived by Lamb (**L1,** 555).

In some applications it is more convenient to work with $\mathbf{x}_1$, the displacement of a fluid element from its equilibrium position, instead of its velocity $\mathbf{v}_1$. To first order in small quantities

$$\mathbf{v}_1 = (\partial \mathbf{x}_1/\partial t) \tag{52.33}$$

hence

$$\mathbf{x}_1(\mathbf{x}_0, t) = \int_0^t \mathbf{v}_1(\mathbf{x}_0, t')\, dt'. \tag{52.34}$$

Thus integrating (52.31) with respect to time we find

$$(\partial^2 \mathbf{x}_1/\partial t^2) = a^2\, \nabla(\nabla \cdot \mathbf{x}_1) + (a^2\, \nabla \cdot \mathbf{x}_1)\nabla \ln \Gamma_1 + (\Gamma_1 - 1)(\nabla \cdot \mathbf{x}_1)\mathbf{g} + \nabla \cdot (\mathbf{g} \cdot \mathbf{x}_1). \tag{52.35}$$

The same approach may be used to rewrite (52.24), (52.25), (52.28), and (52.32) in terms of $\mathbf{x}_1$.

WAVE ENERGY DENSITY AND FLUX

To obtain a wave energy equation we multiply (52.54) by p_1/ρ_0, take the dot product of (52.25) with $\mathbf{v}_1$, multiply (52.27) by $p_1/a^2\rho_0$, and add, obtaining

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 v_1^2+\frac{1}{2}\frac{p_1^2}{a^2\rho_0}\right)=-\nabla\cdot(p_1\mathbf{v}_1)+\rho_1\mathbf{g}\cdot\mathbf{v}_1-\left(\frac{p_1}{a^2\rho_0}\right)\mathbf{v}_1\cdot\nabla p_0$$
$$=-\nabla\cdot(p_1\mathbf{v}_1)+(w_1 g/a^2)(p_1-a^2\rho_1). \qquad (52.36)$$

In (52.36), w_1 denotes the vertical component of $\mathbf{v}_1$.

Now from (52.6) and (52.27) we have

$$(p_1-a^2\rho_1)_{,t}=-w_1[(dp_0/dz)-a^2(d\rho_0/dz)]=-a^2\rho_0\omega_{\mathrm{BV}}^2 w_1/g, \qquad (52.37)$$

which, when integrated with respect to time, yields

$$p_1-a^2\rho_1=-a^2\rho_0\omega_{\mathrm{BV}}^2\zeta_1/g, \qquad (52.38)$$

where ζ_1 is the vertical component of the displacement $\mathbf{x}_1$. Hence the last term in (52.36) equals $-\rho_0\omega_{\mathrm{BV}}^2\zeta_1(\partial\zeta_1/\partial t)$, whence we see that (52.36) can be rewritten as a conservation law

$$(\partial\varepsilon_{\mathrm{w}}/\partial t)+\nabla\cdot\boldsymbol{\phi}_{\mathrm{w}}=0, \qquad (52.39)$$

where the wave energy density is

$$\varepsilon_{\mathrm{w}}=\tfrac{1}{2}\rho_0 v_1^2+\tfrac{1}{2}(p_1^2/a^2\rho_0)+\tfrac{1}{2}\rho_0\omega_{\mathrm{BV}}^2\zeta_1^2, \qquad (52.40)$$

and the wave energy flux is

$$\boldsymbol{\phi}_{\mathrm{w}}=p_1\mathbf{v}_1. \qquad (52.41)$$

We thus obtain the same expression for $\boldsymbol{\phi}_{\mathrm{w}}$ in an acoustic-gravity wave as in a pure acoustic wave [cf. (50.8)]. In contrast, ε_{w} for acoustic-gravity waves contains a *buoyancy energy density* (or *gravitational energy density*) $\frac{1}{2}\rho_0\omega_{\mathrm{BV}}^2\zeta_1^2$ in addition to the kinetic energy and compressional energy densities found previously for pure acoustic waves. The compressional and buoyancy terms both are potential energies for the oscillating fluid parcel. Generalizations of these results to include the effects of magnetic fields for magnetoacoustic-gravity waves are given in (**A2,** 458) and (**B10,** 252).

Although (52.39) is a genuine conservation relation connecting ε_{w} and $\boldsymbol{\phi}_{\mathrm{w}}$, it is not a complete energy equation because a consistent second-order expansion of the nonlinear energy equation contains, in general, nonzero contributions from other second-order quantities such as ρ_2, p_2, etc. For this reason Eckart (**E1,** 53) called ε_{w} the *external energy density* of the wave, remarking that it "has little relation to *e*, the 'internal' energy of thermodynamics". While this terminology is often adopted in the literature, we will not use it because the compressional energy term in ε_{w} does, in fact, originate from the internal energy of the gas [cf. (50.3) to (50.5)]. We merely caution the reader to remember that ε_{w} and $\boldsymbol{\phi}_{\mathrm{w}}$ do not represent the *total* second-order energy density and flux associated with a wave.

53. *Propagation of Acoustic-Gravity Waves in an Isothermal Medium*

For the special case of a static *isothermal atmosphere* it is possible to describe the properties of acoustic-gravity waves analytically in some detail. While this model of the atmosphere is restrictive, it yields considerable physical insight; moreover it is actually not a bad approximation to the temperature-minimum region joining the upper photosphere and the lower chromosphere in the solar atmosphere (see §54).

For simplicity, assume that the material is a perfect gas with constant specific heats. In the ambient atmosphere we then have

$$p_0(z) = p_0(0)e^{-z/H} \tag{53.1a}$$

and

$$\rho_0(z) = \rho_0(0)e^{-z/H} \tag{53.1b}$$

where the *scale height* is

$$H \equiv RT/g = a^2/\gamma g. \tag{53.2}$$

For the special case we are considering, the pressure and density scale heights are equal, which is not true in general (cf. §54).

THE DISPERSION RELATION

Consider monochromatic plane waves of the general form

$$\mathbf{x}_1 = \mathbf{X} \exp\left[i(\omega t - \mathbf{k} \cdot \mathbf{x})\right]. \tag{53.3}$$

We confine attention to steady-state oscillations and therefore take ω to be real. Because the atmosphere is homogeneous in layers perpendicular to the direction of gravity, there is no preferred direction of propagation in the (x, y) plane, hence it suffices to consider propagation in the (x, z) plane only. Thus we take $\mathbf{x}_1 = (\xi_1, 0, \zeta_1)$ or $\mathbf{X} = (X, 0, Z)$ where X and Z are complex constants. Likewise we set $\mathbf{k} = (K_x, 0, K_z)$. In general K_z will be complex because a wave can grow or decay in amplitude as it propagates vertically in a stratified medium; in contrast, the horizontal homogeneity of the ambient atmosphere implies that waves should neither grow nor decay horizontally, hence we can take $K_x \equiv k_x$, a real number.

Substituting (53.3) into the wave equation (52.32), we obtain two homogeneous equations for X and Z, which yield a nontrivial solution only if the determinant of coefficients

$$\begin{vmatrix} a^2k_x^2 - \omega^2 & a^2k_xK_z - igk_x \\ a^2k_xK_z - i(\gamma-1)gk_x & a^2K_z^2 - igK_z - \omega^2 \end{vmatrix} \equiv 0. \tag{53.4}$$

Evaluating (53.4) we obtain the dispersion relation

$$\omega^4 - [a^2(k_x^2 + K_z^2) - i\gamma g K_z]\omega^2 + (\gamma - 1)g^2k_x^2 = 0. \tag{53.5}$$

Demanding that the real and imaginary parts of (53.5) both be zero, we find

$$K_z = k_z + (i/2H) \tag{53.6}$$

where k_z is real. From (53.6) and (53.3) it follows that the amplitudes of $\mathbf{x}_1$ and $\mathbf{v}_1$ both grow as $e^{z/2H}$ with increasing height in the atmosphere. Note in passing that this result and (53.1b) imply that the kinetic energy density $\frac{1}{2}\rho_0 v_1^2$ in the wave is constant with height, a point to which we return later.

Using (53.6) we can rewrite (53.5) as

$$\omega^4 - [\omega_a^2 + a^2(k_x^2 + k_z^2)]\omega^2 + \omega_g^2 a^2 k_x^2 = 0 \tag{53.7}$$

where

$$\omega_a \equiv \gamma g/2a = a/2H \tag{53.8}$$

and

$$\omega_g \equiv (\gamma - 1)^{1/2} g/a = (\gamma - 1)^{1/2} a/\gamma H. \tag{53.9}$$

From (52.6) one sees that ω_g is just the Brunt-Väisälä frequency for an isothermal medium, and is thus relevant to buoyancy oscillations and gravity waves. We will see shortly that ω_a is the minimum frequency for acoustic-wave propagation. Note that

$$\omega_a/\omega_g = \gamma/2(\gamma - 1)^{1/2}, \tag{53.10}$$

hence ω_a is always larger than ω_g for the physically relevant range $1 \leq \gamma \leq \frac{5}{3}$.

THE DIAGNOSTIC DIAGRAM

We can determine a great deal about the behavior of acoustic-gravity waves from an analysis of the dispersion relation. Solving (53.7) for k_z we find

$$k_z^2 = a^{-2}(\omega^2 - \omega_a^2) - (\omega^2 - \omega_g^2)(k_x^2/\omega^2). \tag{53.11}$$

One sees immediately that $k_z^2 < 0$ if $\omega_g \leq \omega \leq \omega_a$, hence no progressive wave can exist in this frequency range, a result that contrasts strongly with that for a homogeneous medium, where there is no restriction on the frequency of propagating waves. Furthermore, we see that as $\omega \rightarrow \infty$, $k^2 = (k_x^2 + k_z^2) \rightarrow \omega^2/a^2$, as in (49.7) for pure acoustic waves; thus waves with $\omega \geq \omega_a$ can be regarded as acoustic waves modified by gravity.

To clarify the picture further, it is helpful to study the domains of wave propagation in the *diagnostic diagram*, a plot of ω versus k_x. We can delineate three distinct domains in the (k_x, ω) plane by finding the *propagation boundary curves* along which $k_z^2 = 0$; these separate regions of real and imaginary vertical wavenumber. Thus setting $k_z^2 = 0$ in (53.11) we have

$$\omega^2 = \tfrac{1}{2}\{\omega_a^2 + a^2 k_x^2 \pm [(\omega_a^2 + a^2 k_x^2)^2 - 4a^2 k_x^2 \omega_g^2]^{1/2}\}, \tag{53.12}$$

which has two branches, as shown in Figure 53.1 for $\gamma = 1.4$. Along the propagation boundary curves we also have

$$k_x^2 = \omega^2(\omega^2 - \omega_a^2)/a^2(\omega^2 - \omega_g^2) \tag{53.13}$$

and

$$v_p^2/a^2 = (\omega^2 - \omega_g^2)/(\omega^2 - \omega_a^2), \tag{53.14}$$

where v_p is the phase speed of the wave.

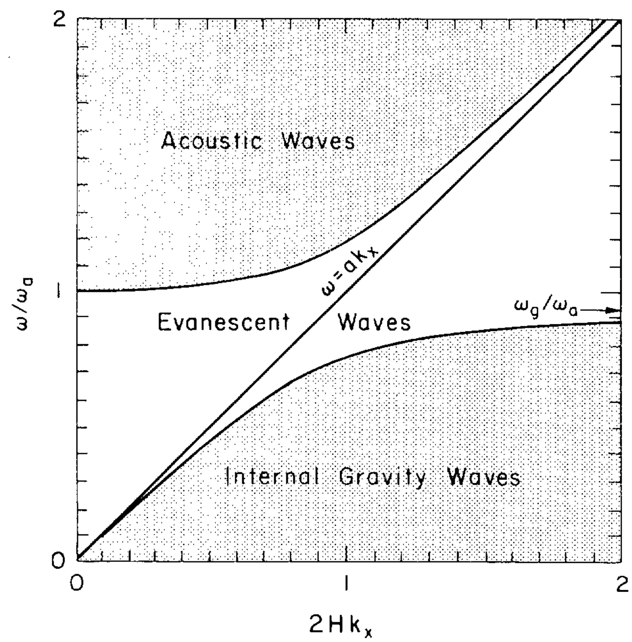

Fig. 53.1 Diagnostic diagram for acoustic-gravity waves in an isothermal atmosphere; $\gamma = 1.4$.

From the remark made above we know that acoustic waves lie in the region above the upper branch of (53.12), and we expect the region below the lower branch to contain gravity waves. Consider first the properties of waves on the upper propagation boundary curve ($\omega \geq \omega_a$). As $k_x \to \infty$, $\omega \to ak_x$, and $v_p \to a$; thus in the high-frequency limit we recover essentially pure acoustic waves, as one would expect because for wavelengths small compared to a scale height the medium is essentially homogeneous (unstratified) over a wavelength. As $k_x \to 0$, $\omega \to \omega_a$, and $v_p \to \infty$. The significance of this result can be appreciated more fully by considering vertically propagating waves.

For vertically propagating waves ($k_x \equiv 0$) we have from (53.11)

$$k_z^2 = (\omega^2 - \omega_a^2)/a^2 \tag{53.15}$$

and

$$v_p^2/a^2 = \omega^2/(\omega^2 - \omega_a^2), \tag{53.16}$$

which show that vertical propagation is possible only if $\omega > \omega_a$; that is, *only* acoustic waves can propagate vertically in a stratified medium. Furthermore, we see that as $\omega \to \omega_a$ there is an *atmospheric resonance* phenomenon, first recognized by Lamb. Because of this resonance, an imposed vertical disturbance at $\omega = \omega_a$ cannot propagate, but rather in effect lifts (or drops) the whole atmosphere *coherently* (which implies that $k_z = 0$ and also

$v_p = \infty$). For vertically propagating acoustic waves, we find from, (53.15) and (53.16), that

$$v_g = (d\omega/dk) = k_z a^2/\omega = a^2/v_p \tag{53.17}$$

or

$$v_p v_g = a^2. \tag{53.18}$$

Notice that because $v_p \to \infty$ as $\omega \to \omega_a$, $v_g \to 0$, hence no energy is propagated by an acoustic disturbance at the atmospheric resonance. Thus ω_a is the *acoustic cutoff frequency, below* which acoustic waves cannot propagate in a stratified atmosphere.

Now consider the lower propagation boundary curve ($\omega \leq \omega_g$). From (53.13) and (53.14) we see that as $k_x \to 0$, $\omega \to (\omega_g/\omega_a)ak_x \to 0$, and $v_p \to (\omega_g/\omega_a)a$. As $k_x \to \infty$, $\omega \to \omega_g$ and $v_p \to 0$. Hence ω_g is a cutoff frequency *above* which gravity waves cannot propagate.

An important property of gravity waves is that they propagate in only a limited range of angles above and below the horizontal. Thus writing $k_x = k \cos \alpha$ we can rearrange (53.7) as

$$\omega^2/a^2k^2 = v_p^2/a^2 = (\omega^2 - \omega_g^2 \cos^2 \alpha)/(\omega^2 - \omega_a^2). \tag{53.19}$$

For an acoustic wave $\omega > \omega_a$, and (53.19) places no restriction on α. But for a gravity wave, $\omega < \omega_g$, the phase velocity is real only if $\omega^2 < \omega_g^2 \cos^2 \alpha$, or if

$$|\alpha| \leq \cos^{-1}(\omega/\omega_g). \tag{53.20}$$

For a fixed k_x, gravity waves can propagate within an ever-increasing range of angles as $\omega \to 0$; but as ω increases toward the lower propagation boundary curve, $k_z \to 0$ hence only horizontal propagation is possible. We can also rewrite (53.7) as

$$\cos^2 \alpha = k_x^2/k^2 = (\omega^2/\omega_g^2) + [\omega^2(\omega_a^2 - \omega^2)/\omega_g^2 a^2 k_x^2], \tag{53.21}$$

which shows that at fixed ω the range of α within which gravity waves can propagate opens from zero on the lower propagation boundary curve to the limiting value $\pm\alpha_{\max} = \cos^{-1}(\omega/\omega_g)$ as $k_x \to \infty$.

The region between the two domains of propagation contains *evanescent waves*. Here k_z is imaginary, hence the wave amplitudes grow or decay exponentially with height. For these waves the pressure perturbation p_1 and the vertical component of $\mathbf{v}_1$ are 90° out of phase, hence they have zero vertical energy flux (though they may transport energy horizontally). Evanescent waves have infinite vertical phase velocity, and therefore represent *standing waves*.

The domain into which a wave specified by a particular (k_x, ω) falls depends on γ, g, and T for the atmosphere. For a given γ and g, (53.8) and (53.9) show that both ω_a and ω_g vary as $T^{-1/2}$, hence both the acoustic-wave and gravity-wave cutoffs decrease with increasing temperature. Propagation boundary curves for $\gamma = \frac{5}{3}$, $g = 3 \times 10^4$ (appropriate to the solar

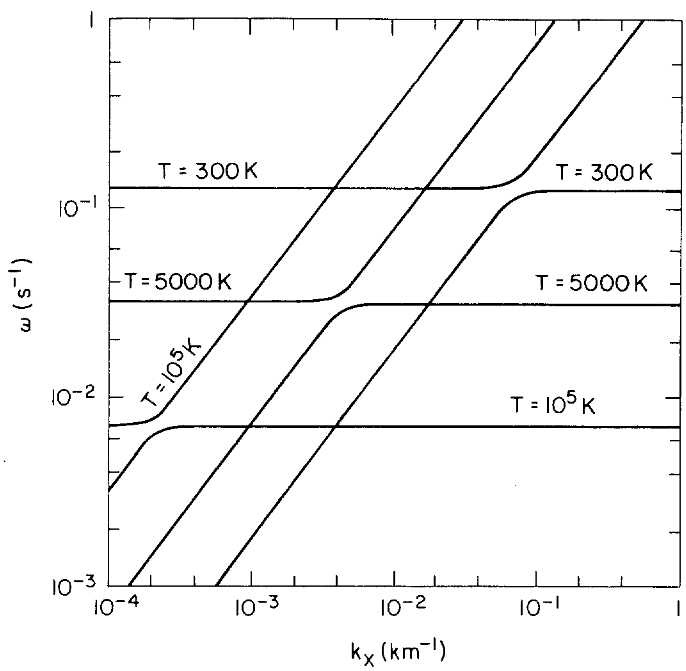

Fig. 53.2 Propagation boundary curves for acoustic-gravity waves in an isothermal atmosphere for several temperatures; $\gamma = \frac{5}{3}$, $g = 3 \times 10^4$.

atmosphere), and several values of T are shown in Figure 53.2. Notice that a wave that can propagate at one temperature may be evanescent at another temperature.

GROUP VELOCITY

Rearranging the dispersion relation as

$$(\omega^2 - \omega_g^2)k_x^2 + \omega^2 k_z^2 = \omega^2(\omega^2 - \omega_a^2)/a^2 \tag{53.22}$$

we see that in wavenumber space the contours of constant ω are conic sections, known as *slowness surfaces*. For acoustic waves, $\omega > \omega_a$, all of the coefficients in (53.22) are positive, hence the contours are ellipses as shown in Figure 53.3a; for gravity waves, $\omega < \omega_g$, the coefficients in (53.22) alternate in sign, hence the contours are hyperbolae, as shown in Figure 53.3b.

The curves shown in Figure 53.3 reveal a great deal about the propagation characteristics of acoustic-gravity waves. When $\omega \gg \omega_a$, the constant-ω ellipses approach the unit circle; as $\omega \rightarrow \omega_a$ they shrink to a point at the origin. For all the ellipses $a^2(k_x^2 + k_z^2)/\omega^2 \leq 1$, hence for acoustic waves the phase speed always exceeds the sound speed. As $\omega \rightarrow \infty$, $v_p \rightarrow a$, and as $\omega \rightarrow \omega_a$, $v_p \rightarrow \infty$. When $\omega \ll \omega_g$ the constant-ω hyperbolae collapse to the asymptotes $ak_x/\omega = \omega_a/\omega_g$. As $\omega \rightarrow \omega_g$ from below, we must have $k_x \gg 1$ to

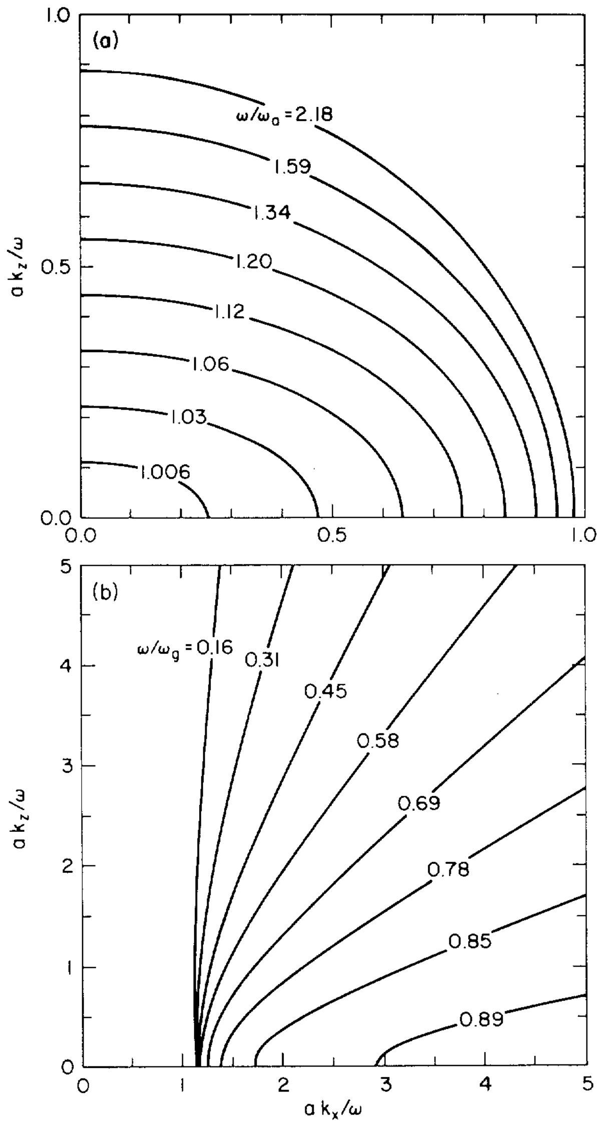

Fig. 53.3 Slowness surfaces in an isothermal atmosphere for (a) acoustic waves, (b) gravity waves.

assure that k_z remains real, hence the vertices of the hyperbolae move toward infinity, and the asymptotes collapse onto the k_x axes. It is evident from Figure 53.3b that $a^2(k_x^2+k_z^2)/\omega^2>1$ for all the hyperbolae, hence for gravity waves the phase speed is always less than the sound speed. Indeed, as $\omega \rightarrow 0$, $v_p \rightarrow (\omega_g/\omega_a)a$, and as $\omega \rightarrow \omega_g$, $v_p \rightarrow 0$. Physically the case $\omega = \omega_g$ corresponds to the stationary buoyancy oscillation described in §52, which does not propagate, hence has $v_p = 0$. All of these results for v_p also follow, of course, from an analysis of (53.19).

As we saw in §49, the direction of phase propagation lies along the wave vector $\mathbf{k}$, and is thus characterized by the angle $\alpha = \cos^{-1}(k_x/k) = \tan^{-1}(k_z/k_x)$. Therefore for a wave with a specified (k_x, ω), the direction of phase propagation lies along the straight line connecting the origin in Figure 53.3 to the curve for the given value of ω at the appropriate value of k_x. On the other hand, the group velocity is given by $\mathbf{v}_g = \nabla_k \omega$, hence it is perpendicular to contours of constant ω. Inspection of Figure 53.3 immediately shows that, for acoustic waves, $\mathbf{v}_p$ and $\mathbf{v}_g$ point almost in the same direction, becoming coincident as $\omega \to \infty$. In contrast, for gravity waves $\mathbf{v}_g$ is usually nearly *perpendicular* to $\mathbf{v}_p$ (becoming exactly perpendicular in the limit $\omega \to 0$); because horizontal phase and energy propagation are in the same direction, this orthogonality implies that vertical phase and energy propagation are oppositely directed. From Figure 53.3b one sees that as $\omega \to 0$, phase propagation in a gravity wave is nearly vertical while energy propagation is essentially horizontal; as $\omega \to \omega_g$, phase propagation becomes more nearly horizontal and energy propagation essentially vertical. Along the $k_z = 0$ axis, corresponding to the lower propagation boundary curve in Figure 53.1, both phase and energy propagation in a gravity wave are horizontal.

We can make the above geometric considerations more quantitative by calculating $(v_g)_i = (\partial\omega/\partial k_i)$ directly from the dispersion relation. Writing $\mathbf{v}_g = (u_g, 0, w_g)$ we find

$$\frac{u_g}{a} = \frac{\omega(\omega^2 - \omega_g^2)ak_x}{\omega^4 - \omega_g^2 a^2 k_x^2} = \frac{(\omega^2 - \omega_g^2)ak_x}{\omega(2\omega^2 - \omega_a^2 - a^2k^2)} \tag{53.23a}$$

and

$$\frac{w_g}{a} = \frac{\omega^3 ak_z}{\omega^4 - \omega_g^2 a^2 k_x^2} = \frac{\omega ak_z}{2\omega^2 - \omega_a^2 - a^2k^2}, \tag{53.23b}$$

which imply a ratio of group speed to sound speed of

$$\begin{aligned}\frac{v_g}{a} &= \frac{\omega[\omega^4 a^2k^2 + \omega_g^2(\omega_g^2 - 2\omega^2)a^2k_x^2]^{1/2}}{\omega^4 - \omega_g^2 a^2 k_x^2} \\ &= \frac{[\omega^4 a^2k^2 + \omega_g^2(\omega_g^2 - 2\omega^2)a^2k_x^2]^{1/2}}{\omega(2\omega^2 - \omega_a^2 - a^2k^2)}.\end{aligned} \tag{53.24}$$

For high-frequency acoustic waves with $\omega \gg \omega_a$ and $\omega^4 \gg \omega_g^2 a^2 k_x^2$ we find

$$u_g/a \approx ak_x/\omega \approx \cos\alpha \tag{53.25a}$$

and

$$w_g/a \approx ak_z/\omega \approx \sin\alpha \tag{53.25b}$$

where we noted that for such waves $\omega \approx ak$. For low-frequency gravity waves with $\omega \ll \omega_g$ and $\omega^4 \ll \omega_g^2 a^2 k_x^2$ we find

$$u_g \approx \omega/k_x \tag{53.26a}$$

and

$$w_g \approx -\omega^3 k_z/\omega_g^2 k_x^2 \approx -\text{sgn}\,(k_z)(\omega/\omega_g)(\omega/k_x) \approx -\text{sgn}\,(k_z)(\omega/\omega_g)u_g \tag{53.26b}$$

because for such waves $|k_z| \approx k \approx (\omega_g/\omega)k_x$. Here $\text{sgn}\,(k_z)$ denotes the algebraic sign of k_z. Note that for $\omega \ll \omega_g$, $|w_g| \ll |u_g|$.

Using the expression for k^2 obtained from (53.19) to eliminate k and k_x from (53.24), we can express v_g in terms of ω, ω_a, ω_g, and α, the angle of phase propagation, as

$$\frac{v_g}{a} = \frac{\{(\omega^2 - \omega_a^2)(\omega^2 - \omega_g^2 \cos^2 \alpha)[\omega^4 + \omega_g^2(\omega_g^2 - 2\omega^2) \cos^2 \alpha]\}^{1/2}}{\omega^4 + \omega_g^2(\omega_a^2 - 2\omega^2) \cos^2 \alpha}. \tag{53.27}$$

Notice that $v_g = 0$ for gravity waves propagating at the limiting angle $\alpha_{\max} = \cos^{-1}(\omega/\omega_g)$.

Alternatively, if β is the angle between $\mathbf{v}_g$ and the horizontal, then $u_g = v_g \cos \alpha$ and $w_g = v_g \sin \alpha$, hence from (53.23)

$$\tan \beta = [\omega^2/(\omega^2 - \omega_g^2)] \tan \alpha. \tag{53.28}$$

Using (53.28) in (53.27) one finds

$$\frac{v_g}{a} = \frac{[(\omega^2 - \omega_g^2)^3(\omega^2 - \omega_a^2)(\omega^2 - \omega_g^2 \sin^2 \beta)]^{1/2}}{\omega[(\omega^2 - \omega_g^2)^2 + \omega_g^2(\omega_a^2 - \omega_g^2) \cos^2 \beta]}, \tag{53.29}$$

which shows that energy propagation for gravity waves can occur only for angles less than $\pm\beta_{\max} = \sin^{-1}(\omega/\omega_g)$ away from the horizontal.

POLARIZATION RELATIONS

To obtain a more complete description of an acoustic-gravity wave write the fluctuations in (ρ, p, T) and $\mathbf{v}_1 \equiv (u_1, 0, w_1)$ as

$$\frac{\rho_1}{\rho_0 R} = \frac{p_1}{\rho_0 P} = \frac{T_1}{T_0 \Theta} = \frac{u_1}{U} = \frac{w_1}{W} = \frac{\xi_1}{X} = \frac{\zeta_1}{Z} = e^{z/2H} e^{i(\omega t - k_x x - k_z z)} \tag{53.30}$$

where the amplitudes R, P, Θ, U, W, X, and Z are complex constants. Substituting these representations into the linearized fluid equations (52.24), (52.25), and (52.27) we obtain

$$i\omega R - ik_x U - [(1/2H) + ik_z]W = 0, \tag{53.31a}$$

$$-ik_x P + i\omega U = 0, \tag{53.31b}$$

$$gR - [(1/2H) + ik_z]P + i\omega W = 0, \tag{53.31c}$$

and

$$-ia^2\omega R + i\omega P + [(\gamma - 1)a^2/\gamma H]W = 0. \tag{53.31d}$$

For these equations to have a nontrivial solution, the determinant of the coefficients must be zero; enforcing this requirement we recover the dispersion relation (53.7).

The system (53.31) also yields *polarization relations* that specify the relative amplitudes and phases among the perturbed variables. One finds

$$R = \frac{\omega}{(\omega^2 - a^2k_x^2)}\left\{k_z + \frac{i}{H}\left[\frac{(\gamma-1)a^2k_x^2}{\gamma\omega^2} - \frac{1}{2}\right]\right\}W, \tag{53.32a}$$

$$P = \frac{a^2\omega}{(\omega^2 - a^2k_x^2)}\left[k_z + \frac{i}{H}\left(\frac{\gamma-2}{2\gamma}\right)\right]W, \tag{53.32b}$$

and

$$U = \frac{a^2k_x}{(\omega^2 - a^2k_x^2)}\left[k_z + \frac{i}{H}\left(\frac{\gamma-2}{2\gamma}\right)\right]W; \tag{53.32c}$$

furthermore, from the perfect gas law $(T_1/T_0) = (p_1/p_0) - (\rho_1/\rho_0)$, hence

$$\Theta = \frac{(\gamma-1)\omega}{(\omega^2 - a^2k_x^2)}\left[k_z + \frac{i}{H}\left(\frac{1}{2} - \frac{a^2k_x^2}{\gamma\omega^2}\right)\right]W. \tag{53.32d}$$

Finally, by integrating with respect to time we find $X = U/i\omega$ and $Z = W/i\omega$.

The relative amplitudes of the wave perturbations are thus

$$\left|\frac{\rho_1}{\rho_0}\right| = \frac{\omega a k_z}{|\omega^2 - a^2k_x^2|}\left\{1 + \left(\frac{1}{2k_zH}\right)^2\left[\frac{2(\gamma-1)a^2k_x^2}{\gamma\omega^2} - 1\right]^2\right\}^{1/2}\left|\frac{w_1}{a}\right|, \tag{53.33a}$$

$$\left|\frac{p_1}{p_0}\right| = \frac{\gamma\omega a k_z}{|\omega^2 - a^2k_x^2|}\left[1 + \left(\frac{\gamma-2}{2\gamma}\right)^2\left(\frac{1}{2k_zH}\right)^2\right]^{1/2}\left|\frac{w_1}{a}\right|, \tag{53.33b}$$

$$\left|\frac{u_1}{a}\right| = \frac{a^2k_xk_z}{|\omega^2 - a^2k_x^2|}\left[1 + \left(\frac{\gamma-2}{2\gamma}\right)^2\left(\frac{1}{2k_zH}\right)^2\right]^{1/2}\left|\frac{w_1}{a}\right|, \tag{53.33c}$$

and

$$\left|\frac{T_1}{T_0}\right| = \frac{(\gamma-1)\omega a k_z}{|\omega^2 - a^2k_x^2|}\left[1 + \left(\frac{1}{2k_zH}\right)^2\left(1 - \frac{2a^2k_x^2}{\gamma\omega^2}\right)^2\right]^{1/2}\left|\frac{w_1}{a}\right|. \tag{53.33d}$$

In the high-frequency limit, $\omega \gg \omega_a$, and for nearly vertical phase propagation (so that $\omega \gg ak_x$), $(2k_zH)^{-1} \approx \omega_a/\omega \ll 1$, and the perturbation amplitudes limit to

$$\begin{aligned}&|\rho_1/\rho_0| : |p_1/p_0| : |T_1/T_0| : |u_1/a| : |w_1/a| \\ &\qquad = \sin\alpha : \gamma\sin\alpha : (\gamma-1)\sin\alpha : \sin\alpha\cos\alpha : 1.\end{aligned} \tag{53.34}$$

Here $\sin\alpha = k_z/k$. Notice that the relative amplitudes are the same as for a pure acoustic wave, and do not depend on k or ω. The angular factors merely describe the orientation of $\mathbf{k}$ and have no further significance.

In the low-frequency limit, $\omega \ll \omega_g$ and $\omega^2 \ll a^2k_x^2$, we have $k_z \approx (\omega_g/\omega)k_x$, and the perturbation amplitudes limit to

$$\begin{aligned}&|\rho_1/\rho_0| : |p_1/p_0| : |T_1/T_0| : |u_1/a| : |w_1/a| \\ &\qquad = (\gamma-1)^{1/2}(\omega_g/\omega) : \gamma\omega_g/ak_x : (\gamma-1)^{1/2}(\omega_g/\omega) : \omega_g/\omega : 1.\end{aligned} \tag{53.35}$$

Thus for low-frequency gravity waves $|\rho_1/\rho_0| = |T_1/T_0|$, and we see that $|\rho_1/\rho_0|$, $|T_1/T_0|$, and $|u_1/a|$ can become arbitrarily large relative to $|w_1/a|$ as $\omega \to 0$; in contrast, the fractional pressure fluctuation can be either large or small, depending on the relative size of ak_x and ω_g. Essentially the pressure perturbation drives the horizontal flow and is large for gravity waves with large horizontal wavelengths.

To determine relative phases of the perturbations, note that the phase shift δ_{AB} between any two complex quantities A and B is given by

$$\tan \delta_{AB} = \mathrm{Im}\,(A/B)/\mathrm{Re}\,(A/B). \tag{53.36}$$

In using (53.36) one must be careful about quadrants. A positive (negative) phase shift implies that A leads (lags) B in time by $\delta_{AB}/2\pi$ periods. From (53.36) and (53.32) we find

$$\tan \delta_{PW} = (2k_zH)^{-1}[(\gamma-2)/\gamma], \tag{53.37a}$$

$$\tan \delta_{RW} = (2k_zH)^{-1}\{[2(\gamma-1)a^2k_x^2/\gamma\omega^2]-1\}, \tag{53.37b}$$

and

$$\tan \delta_{\Theta W} = (2k_zH)^{-1}[1-(2a^2k_x^2/\gamma\omega^2)]. \tag{53.37c}$$

Note that for an adiabatic wave $\delta_{PU} \equiv 0$, that is, the pressure and the horizontal velocity fluctuations are always exactly in phase; therefore the horizontal wave energy flux propagates in the same direction as the horizontal phase velocity and is nonzero unless $k_x \equiv 0$.

Now consider the high- and low-frequency limits of (53.37); for definiteness, assume upward propagating waves ($k_z > 0$). For high-frequency acoustic waves in which $\omega^2 \gg a^2k_x^2$ we see from (53.32) that in general $-90° \le \delta_{RW} \le 0°$, $-90° \le \delta_{PW} \le 0°$, and $0° \le \delta_{\Theta W} \le 90°$. For waves with $\omega \gg \omega_a$ and $ak_x \ll \omega$, $(2k_zH)^{-1} \approx \omega_a/\omega \ll 1$, and all the phase shifts are small, of order ω_a/ω. T_1 leads, and ρ_1 and p_1 lag, w_1 slightly. The vertical energy flux propagates in the same direction as the vertical phase velocity. For waves near the propagation boundary curve, where $k_z \to 0$, $\delta_{PW} \to -90°$, and therefore the vertical energy flux vanishes.

For low-frequency gravity waves in which $\omega^2 \ll a^2k_x^2$ and $\omega^2 \ll \omega_g^2$ we see from (53.32) that in general $-180° \le \delta_{RW} \le -90°$, $90° \le \delta_{PW} \le 180°$, and $90° \le \delta_{\Theta W} \le 180°$. When $\omega \ll \omega_g$ the ω^{-2} terms in (53.37) dominate, and we see that $\delta_{RW} \to -90°$ and $\delta_{\Theta W} \to 90°$, in agreement with physical descriptions of buoyancy oscillations given at the beginning of this section. Furthermore, for such waves $2k_zH \approx \omega\omega_a/\omega_g ak_x$, hence $\delta_{PW} \to 180°$, which implies downward energy propagation when there is upward phase propagation, and vice versa.

WAVE ENERGY DENSITY AND FLUX

Using (50.15) we find the time-averaged kinetic energy density in an acoustic-gravity wave is

$$\tfrac{1}{4}\rho_0 e^{z/H}(UU^* + WW^*) = \rho_0(0)(\omega^4 - \omega_g^2a^2k_x^2)WW^*/4\omega^2(\omega^2 - a^2k_x^2), \tag{53.38}$$

the time-averaged compressional energy density is

$$(\rho_0/4a^2)e^{z/H}PP^* = \rho_0(0)(\omega^2-\omega_g^2)WW^*/4(\omega^2-a^2k_x^2), \qquad (53.39)$$

and the time-averaged buoyancy energy density is

$$\tfrac{1}{4}\rho_0\omega_g^2\zeta_1\zeta_1^* = \tfrac{1}{4}\rho_0(0)(\omega_g/\omega)^2WW^*, \qquad (53.40)$$

hence the time-averaged wave energy density is

$$\varepsilon_w = \rho_0(0)(2\omega^2-\omega_a^2-a^2k^2)WW^*/2(\omega^2-a^2k_x^2). \qquad (53.41)$$

Note that the sum of the compressional and buoyancy energies exactly equals the kinetic energy, as one would expect from the virial theorem.

The time-averaged energy flux is

$$(\phi_w)_x = \tfrac{1}{4}(p_1u_1^*+p_1^*u_1) = \rho_0(0)a^2(\omega^2-\omega_g^2)k_xWW^*/2\omega(\omega^2-a^2k_x^2) \qquad (53.42a)$$

and

$$(\phi_w)_z = \tfrac{1}{4}(p_1w_1^*+p_1^*w_1) = \rho_0(0)a^2\omega k_zWW^*/2(\omega^2-a^2k_x^2). \qquad (53.42b)$$

Comparing (53.41) and (53.23) with (53.42) we see that $\boldsymbol{\phi}_w = \varepsilon_w\mathbf{v}_g$, as expected. Notice that all components of ε_w and $\boldsymbol{\phi}_w$ are independent of z, hence the wave energy density and flux of adiabatic waves are constant with height in an isothermal atmosphere.

From (53.38) and (53.39) one sees that

$$(\varepsilon_w)_b = [\omega_g^2(\omega^2-a^2k_x^2)/(\omega^4-\omega_g^2a^2k_x^2)](\varepsilon_w)_k \qquad (53.43a)$$

and

$$(\varepsilon_w)_c = [\omega^2(\omega^2-\omega_g^2)/(\omega^4-\omega_g^2a^2k_x^2)](\varepsilon_w)_k \qquad (53.43b)$$

where the subscripts denote "buoyancy", "compressional", and "kinetic". Thus for acoustic waves, $\omega^2 \gg \omega_g^2$,

$$(\varepsilon_w)_b \to (\omega_g^2/\omega^2)(\varepsilon_w)_k \qquad (53.44a)$$

and

$$(\varepsilon_w)_c \to [1-(\omega_g^2/\omega^2)](\varepsilon_w)_k \qquad (53.44b)$$

for small k_x, where $\omega^2 \gg a^2k_x^2$; whereas for large k_x, where $a^2k_x^2 \approx \omega^2-\omega_a^2$,

$$(\varepsilon_w)_b \to (\omega_a^2\omega_g^2/\omega^4)(\varepsilon_w)_k \qquad (53.45a)$$

and

$$(\varepsilon_w)_c \to [1-(\omega_a^2\omega_g^2/\omega^4)](\varepsilon_w)_k. \qquad (53.45b)$$

For gravity waves, $\omega^2 \ll \omega_g^2$, we find

$$(\varepsilon_w)_b \to [(a^2k_x^2-\omega^2)/a^2k_x^2](\varepsilon_w)_k \qquad (53.46a)$$

and

$$(\varepsilon_w)_c \to (\omega^2/a^2k_x^2)(\varepsilon_w)_k. \qquad (53.46b)$$

Thus for acoustic waves nearly all the potential energy is compressional, whereas for gravity waves it is nearly all gravitational.

54. Propagation of Acoustic-Gravity Waves in a Stellar Atmosphere

We now consider the propagation of acoustic-gravity waves in a stratified medium in which the temperature and ionization state of the medium vary with height, for example, the atmospheres of the Sun and other stars. Strong impetus to the development of acoustic-gravity wave theory in astrophysics was given by the discovery of prominent oscillatory motions in the solar atmosphere (**L6**), (**N1**). Later research has shown that the observed motions are the evanescent tails of standing, gravity modified, acoustic eigenmodes, trapped in a subphotospheric resonant cavity. Strictly speaking the theory developed here is not applicable to these trapped modes because we omit discussion of the effects of boundary conditions; even so, it is often instructive to consider them simply as evanescent acoustic-gravity waves that happen to exist for a discrete set of (ω, k_x) combinations. Further motivation for studying acoustic-gravity waves is provided by observations of both propagating and trapped acoustic waves in the solar atmosphere, and by the existence of a substantial nonthermal broadening of solar spectrum lines, much of which is thought to be caused by unresolved, small-amplitude wave motions.

To study acoustic-gravity wave propagation in realistic models of stellar atmospheres we must understand the phenomena of wave refraction and reflection and their implications for wave tunneling and trapping. Furthermore, we must account for ionization effects and temperature gradients in (1) calculation of the sound speed, scale height, and Brunt-Väisälä frequency; (2) formulation of the wave equation; and (3) the expression relating the temperature perturbation to the vertical velocity.

REFLECTION, REFRACTION, TUNNELING, AND TRAPPING

In a nonisothermal atmosphere, propagating waves experience *refraction* and *partial reflection.* If they encounter a semi-infinite region in which they are evanescent, they can be *totally reflected* with no transmission of energy beyond the point of reflection. If, however, the evanescent layer is finite, some fraction of the incident wave energy may leak through the region by *tunneling,* and the wave continues to propagate, with reduced amplitude, on the other side of the barrier. We discuss these phenomena first for discontinuous changes in atmospheric properties at definite boundaries, and then for a slowly varying atmosphere, where we can apply the WKB approximation.

(*a*) *Reflection and Refraction at an Interface* When a wave encounters a discontinuity in material properties, (1) the propagation vector **k** changes in both direction and magnitude; (2) the absolute and relative amplitudes of the wave perturbations change; and (3) phase differences between perturbations may change.

To illustrate the basic properties of refraction and partial reflection we consider a plane, monochromatic, acoustic wave propagating from one homogeneous medium, A, with sound speed a_A, into a second homogeneous medium, B, with sound speed a_B. The interface between the two media is the plane $z=z_1$; A is the region $z \leq z_1$ and B is $z \geq z_1$. Choose the x axis such that $\mathbf{k}=(k_x, 0, k_z)$. Continuity at the interface requires that k_x and ω be the same in both media, whereas k_z, given by the dispersion relation $k_z^2 = a^2\omega^2 - k_x^2$, changes. Therefore the angle of propagation $\theta = \tan^{-1}(k_x/k_z)$ between $\mathbf{k}$ and the z axis changes. We readily find that

$$\sin\theta_A/\sin\theta_B = (k_x/k_A)/(k_x/k_B) = a_A/a_B. \tag{54.1}$$

Thus if $a_B > a_A$ (i.e., $T_B > T_A$) the wave refracts away from the vertical, and refracts towards it if $a_B < a_A$.

The behavior of a wave incident on a discontinuity follows from continuity conditions at the interface. First, pressure equilibrium dictates that $p_{0A}(z_1) = p_{0B}(z_1)$ and $p_{1A}(z_1) = p_{1B}(z_1)$. The former condition implies a density discontinuity given by $(\rho_{0A}/\rho_{0B}) = (T_{0B}\mu_A/T_{0A}\mu_B)$ where μ is the mean molecular weight. Second, w_1 must be continuous to avoid a vacuum or interpenetration of material elements. Third, the energy flux must be continuous because there can be no sources or sinks in an interface of zero thickness. Continuity of p_1, w_1, and $\boldsymbol{\phi}_w$ provide three independent equations that determine the amplitudes of the reflected and refracted waves, and the phase shift of the reflected wave, in terms of the amplitude of the incident wave.

Following (53.30) we write w_1 in medium A as

$$w_{1A} = W_A^+ e^{i(\omega t - k_x x)} e^{-ik_{zA}(z-z_0)} + W_A^- e^{i(\omega t - k_x x)} e^{ik_{zA}(z-z_0)}, \tag{54.2}$$

where W_A^+ and W_A^- are complex amplitude functions for the incident and reflected waves, respectively. We set $\exp(z/2H) \equiv 1$ because H is infinite (each medium is homogeneous). Similarly

$$w_{1B} = W_B^+ e^{i(\omega t - k_x x)} e^{-ik_{zB}(z-z_1)}. \tag{54.3}$$

The pressure perturbation follows from (53.30) and (53.31c) with $g=0$ and $H=\infty$, namely $p_1^{\pm} = [\rho_0\omega/(\pm k_z)]W^{\pm}$. Hence

$$p_{1A} = (\rho_{0A}\omega/k_{zA})(W_A^+ - W_A^-) \tag{54.4a}$$

and

$$p_{1B} = (\rho_{0B}\omega/k_{zB})W_B^+. \tag{54.4b}$$

To simplify the notation, write the amplitude functions $W^{\pm}$ in terms of real (positive) amplitudes and real phases: $W_A^+ \equiv A_1 e^{i\delta_A^+}$, $W_A^- \equiv A_2 e^{i\delta_A^-}$, and $W_B^+ \equiv B e^{i\delta_B^+}$. Then factoring W_A^+ out of both w_{1A} and w_{1B}, and defining $A \equiv A_2/A_1$, $B \equiv B_1/A_1$, $\delta_A \equiv \delta_A^- + 2k_{zA}(z_1 - z_0)$, and $\delta_B \equiv \delta_B^+ - \delta_A^-$ we can write

$$w_{1A}(z_1)=A_1e^{i\delta_A^+}e^{i[\omega t-k_xx-k_z(z_1-z_0)]}(1+Ae^{i\delta_A}) \quad (54.5a)$$

and

$$w_{1B}(z_1)=A_1e^{i\delta_A^+}e^{i(\omega t-k_xx)}Be^{i\delta_B}. \quad (54.5b)$$

In terms of the amplitudes and phases just defined and the parameter

$$r\equiv\rho_{0B}k_{zA}/\rho_{0A}k_{zB} \quad (54.6)$$

the three continuity conditions

$$w_{1A}(z_1)=w_{1B}(z_1), \quad (54.7a)$$

$$p_{1A}(z_1)=p_{1B}(z_1), \quad (54.7b)$$

and

$$\tfrac{1}{4}(p_{1A}^*w_{1A}+p_{1A}w_{1A}^*)=\tfrac{1}{4}(p_{1B}^*w_{1B}+p_{1B}w_{1B}^*) \quad (54.7c)$$

can be written

$$e^{-ik_{zA}(z_1-z_0)}(1+Ae^{i\delta_A})=Be^{i\delta_B}, \quad (54.8a)$$

$$e^{-ik_{zA}(z_1-z_0)}(1-Ae^{i\delta_A})=rBe^{i\delta_B}, \quad (54.8b)$$

and

$$1-A^2=rB^2. \quad (54.8c)$$

Multiplying (54.8a) and (54.8b) by their complex conjugates we have

$$1+A^2+2A\cos\delta_A=B^2, \quad (54.9a)$$

$$1+A^2-2A\cos\delta_A=r^2B^2, \quad (54.9b)$$

and

$$1-A^2=rB^2, \quad (54.9c)$$

Solving (54.14) we find that the square of the ratio of the reflected wave amplitude to the incident wave amplitude is

$$A^2=|W_A^-|^2/|W_A^+|^2=(1-r)^2/(1+r)^2<1, \quad (54.10)$$

which is sometimes called the *reflectance* (**L2**). When $r\ll 1$ or $r\gg 1$, $A^2\approx 1$ and the wave is almost totally reflected; as the discontinuity becomes small, $r\rightarrow 1$, $A^2\rightarrow 0$, and the wave is almost totally transmitted. Using (54.1) and (54.6) we can write the reflectance in terms of material properties and the angle of incidence as

$$A^2=\left[\frac{\rho_{0B}a_A\cos\theta_A-\rho_{0A}(a_A^2-a_B^2\sin^2\theta_A)^{1/2}}{\rho_{0B}a_B\cos\theta_A-\rho_{0A}(a_A^2-a_B^2\sin^2\theta_A)^{1/2}}\right]^2. \quad (54.11)$$

From (54.9), the ratio of the amplitude of the refracted (transmitted) wave to the amplitude of the incident wave is

$$B=|W_B^+|/|W_A^+|=2/(1+r), \quad (54.12)$$

which is greater than or less than unity depending on whether r is less than or greater than unity. This result is not, as it might seem, paradoxical; when the amplitude of w_1 in the transmitted wave exceeds that in the incident wave ($r<1$, $B>1$), there is a compensating decrease in the amplitude of p_1 such that the wave energy fluxes in A and B are exactly equal. In fact, (54.11) and (54.12) imply

$$\phi_{wA} = \tfrac{1}{2}(\omega\rho_{0A}/k_{zA})A_1^2(1-A^2) = (\omega\rho_{0A}/k_{zA})A_1^2[2r/(1+r)^2] \quad (54.13)$$

and

$$\phi_{wB} = \tfrac{1}{2}(\omega\rho_{0B}/k_{zB})A_1^2B^2 = (\omega\rho_{0B}/k_{zB})A_1^2[2/(1+r)^2], \quad (54.14)$$

which are seen to be equal by recalling the definition of r. Note that the flux is very small whenever the discontinuity is large: when $r \gg 1$, $2r/(1+r)^2 \to 2/r \ll 1$, and when $r \ll 1$, $2r/(1+r)^2 \to 2r \ll 1$. Thus little energy is transmitted across the boundary when $a_A \gg a_B$ or $a_B \gg a_A$.

(*b*) *An Interface Between Evanescent and Propagation Regions* The foregoing analysis assumes that the wave is propagating ($k_z^2>0$) in both regions. Consider now the case where the wave is evanescent either in A ($k_{zA}^2<0$) or in B ($k_{zB}^2<0$); the case of evanescence in both is uninteresting. In an evanescent wave, the energy density decreases exponentially with distance into the region of evanescence. The wavenumber k_z is a pure imaginary, hence ik_z is real, and the z-momentum equation shows that p_1 is 90° out of phase with w_1; therefore the wave carries no energy flux.

Suppose first that $k_{zB}^2<0$, and write $\kappa_B = ik_{zB}$. Assume that medium B is semi-infinite so that we have an upward-decaying solution $w_{1B} = W_B^+ \exp[i(\omega t - k_x x)] \exp[-\kappa_B(z-z_1)]$. The equations corresponding to (54.9) then become

$$1 + A^2 + 2A\cos\delta_A = B^2, \quad (54.15a)$$

$$1 + A^2 - 2A\cos\delta_A = r^2B^2, \quad (54.15b)$$

and

$$(\omega\rho_{0A}/k_{zA})A_1^2(1-A^2) = 0, \quad (54.15c)$$

where now

$$r \equiv \rho_{0B}k_{zA}/\rho_{0A}\kappa_B. \quad (54.16)$$

These yield

$$A = 1, \quad (54.17a)$$

$$B^2 = 4/(1+r^2), \quad (54.17b)$$

$$\cos\delta_A = (1-r^2)/(1+r^2), \quad (54.17c)$$

and $\phi_{wA} = \phi_{wB} = 0$. We thus have total reflection in medium A, and excite an evanescent disturbance in medium B.

Suppose now that the wave is evanescent in a finite region A ($k_{zA}^2<0$), but can propagate in medium B, which is semi-infinite. In medium A we can now admit both exponentially decaying and growing solutions. Defining $\kappa_A \equiv ik_{zA}$ we can write w_{1A} with respect to some reference level z_0 as

$$w_{1A} = A_1' e^{i\delta}[e^{-\kappa_A(z-z_0)} + A' e^{\kappa_A(z-z_0)} e^{i\delta_A}] \tag{54.18}$$

or

$$w_{1A}(z_1) = A_1' e^{i\delta}(e^{-\Delta} + A' e^{\Delta} e^{i\delta_A}) \tag{54.19}$$

where $\Delta \equiv \kappa_A(z_1 - z_0)$. To write the continuity conditions at z_1 we define $A_1 \equiv A_1' e^{-\Delta}$, $A \equiv A' e^{2\Delta}$, and $B \equiv B_1/A_1 = B_1/(A_1' e^{-\Delta})$. Then at z_1, $w_{1A}(z_1) = A_1 e^{i\delta}(1 + A e^{i\delta_A})$ and $w_{1B}(z_1) = A_1 B e^{i\delta_B}$, and continuity of w_1, p_1, and $\boldsymbol{\phi}_w$ imply

$$1 + A^2 + 2A \cos \delta_A = B^2, \tag{54.20a}$$

$$1 + A^2 - 2A \cos \delta_A = r^2 B^2, \tag{54.20b}$$

and

$$2A \sin \delta_A = rB^2 \tag{54.20c}$$

where now

$$r \equiv \rho_{0B}\kappa_A/\rho_{0A}k_{zB}. \tag{54.21}$$

Solving these equations we again find

$$A = 1 \tag{54.22a}$$

and

$$B^2 = 4/(1 + r^2). \tag{54.22b}$$

The phase shifts are now

$$\tan \delta_A = 2r/(1 - r^2) \tag{54.23a}$$

and

$$\tan \delta_B = r. \tag{54.23b}$$

Interference between the growing and decaying evanescent waves in medium A produces a phase lag between w_{1A} and p_{1A} that is not exactly 90°. One finds that the energy flux in region A is

$$\phi_{wA} = (2\omega\rho_{0B}/k_{zB}) A_1' e^{-\Delta}/(1 + r^2), \tag{54.24}$$

which decreases exponentially with increasing Δ.

(*c*) *Tunneling* Wave tunneling, which is related to the second case just described, occurs when a wave that is propagating in medium A encounters a finite layer B in which it is evanescent (with growing and decaying solutions), and then emerges into a layer C (perhaps semi-infinite) in which it can again freely propagate ($k_{zC}^2 > 0$). In layer A we have both an incident and reflected wave; in layer C we have only an outward-propagating transmitted wave.

If the interface between A and B is at z_1, and that between B and C is at z_2, the thickness of layer B is $z_2 - z_1$. The vertical velocities in the three regions are

$$w_{1A} = A_1 e^{i(\omega t - k_x x)} e^{ik_{zA}(z-z_0)}(1 + Ae^{i\delta_A}) \tag{54.25a}$$

$$w_{1B} = A_1 e^{i(\omega t - k_x x)}[B_1 e^{-\kappa_B(z-z_1)} e^{i\delta_B^-} + B^- e^{\kappa_B(z-z_1)} e^{i\delta_B^-}], \tag{54.25b}$$

and

$$w_{1C} = A_1 e^{i(\omega t - k_x x)} C e^{ik_{zC}(z-z_2)} e^{i\delta_C^+}. \tag{54.25c}$$

The matching conditions at z_1 and z_2 yield six equations in A_1, B_1, B_2, C, δ_A, and $\delta_B \equiv \delta_B^- - \delta_B^+$. At z_1 we have

$$1 + A^2 + 2A \cos \delta_A = B_1^2 + B_2^2 + 2B_1 B_2 \cos \delta_B, \tag{54.26a}$$

$$1 + A^2 - 2A \cos \delta_A = r_1^2(B_1^2 + B_2^2 - B_1 B_2 \cos \delta_B), \tag{54.26b}$$

and

$$1 - A^2 = 2r_1 B_1 B_2 \sin \delta_B. \tag{54.26c}$$

At z_2, defining $b_1 \equiv B_1 e^{-\Delta}$, $b_2 \equiv B_2 e^{\Delta}$, and $\Delta \equiv \kappa_B(z_2 - z_1)$ we have

$$b_1^2 + b_2^2 + 2b_1 b_2 \cos \delta_B = C^2, \tag{54.27a}$$

$$b_1^2 + b_2^2 - 2b_1 b_2 \cos \delta_B = r_2^2 C^2, \tag{54.27b}$$

and

$$2b_1 b_2 \sin \delta_B = r_2 C^2. \tag{54.27c}$$

Here

$$r_1 \equiv \rho_{0B} k_{zA} / \rho_{0A} \kappa_B \tag{54.28a}$$

and

$$r_2 \equiv \rho_{0C} \kappa_B / \rho_{0B} k_{zC}. \tag{54.28b}$$

The solution at z_2 is $b_1 = b_2$, $C = 4b_1/(1 + r_2^2)$, $\tan \delta_B = 2r_2/(1 - r_2^2)$, and $\tan \delta_C = r_2$, which then gives at z_1:

$$1 + A^2 + 2A \cos \delta_A = b_1^2 K_1, \tag{54.29a}$$

$$1 + A^2 - 2A \cos \delta_A = r_1^2 b_1^2 K_2, \tag{54.29b}$$

and

$$1 - A^2 = 2b_1^2 K_3, \tag{54.29c}$$

where

$$K_1 \equiv 2\{\cosh 2\Delta + [(1 - r_2^2)/(1 + r_2^2)]\}, \tag{54.30a}$$

$$K_2 \equiv 2\{\cosh 2\Delta - [(1 - r_2^2)/(1 + r_2^2)]\}, \tag{54.30b}$$

and

$$K_3 \equiv 2r_2/(1 + r_2^2). \tag{54.30c}$$

Solving (54.29) we find

$$b_1^2 = 4(K_1 + r_1^2 K_2 + 4r_1 K_3)^{-1} \tag{54.31}$$

and

$$A^2 = \frac{(K_1 + r_1^2 K_2 - 4r_1 K_3)}{(K_1 + r_1^2 K_2 + 4r_1 K_3)} \tag{54.32}$$

or

$$A^2 = \frac{(1+r_1^2)(1+r_2^2)\cosh 2\Delta + (1-r_1^2)(1-r_2^2) - 4r_1r_2}{(1+r_1^2)(1+r_2^2)\cosh 2\Delta + (1-r_1^2)(1-r_2^2) + 4r_1r_2}. \tag{54.33}$$

Note that when $\Delta \to 0$, so that the evanescent region vanishes, $A^2 = (1-r_1r_2)^2/(1+r_1r_2)^2$, which from the definitions of r_1 and r_2 is (54.10) with $r \equiv \rho_{0C}k_{zA}/\rho_{0A}k_{zC}$. As Δ increases, $\cosh 2\Delta$ increases exponentially and A^2 rapidly approaches unity, that is, the total reflection limit.

The energy flux is given by

$$\begin{aligned}\phi_w &= \frac{A_1^2}{2}\left(\frac{\omega\rho_{0A}}{k_{zA}}\right)(1-A^2)\\ &= \frac{\omega\rho_{0A}A_1^2}{k_{zA}}\left[\frac{4r_1r_2}{(1+r_1^2)(1+r_2^2)\cosh 2\Delta + (1-r_1^2)(1-r_2^2) + 4r_1r_2}\right]\end{aligned} \tag{54.34}$$

which diminishes rapidly with increasing Δ because of the factor $\cosh 2\Delta$ in the denominator. For fairly small values of $\Delta = \kappa_B(z_2 - z_1)$, which occurs when the wavelength of the disturbance is large compared to the thickness of the evanescent zone, a nonnegligible fraction of the energy flux can leak through layer B, appearing in C as a propagating wave. This process is closely analogous to quantum mechanical tunneling, and is observed to occur for various types of waves in both the Earth's and the Sun's atmospheres.

The mathematical description of refraction and partial reflection of acoustic-gravity waves incident on a discontinuity is considerably more complicated, and will not be included here as it adds little physical insight.

(*d*) *Trapping* Another interesting effect of atmospheric structure is wave trapping. Here one has a region in which waves can freely propagate, bounded on both top and bottom by layers in which the waves are nonpropagating. The waves are thus totally reflected at the boundaries of the propagating layer, hence two waves with the same ω and k_x having equal but opposite k_z's will interfere destructively unless their phases are such that they form a *standing wave.*

Standing wave conditions are easiest to describe for waves confined in a *cavity* between two rigid boundaries. Consider pure acoustic waves with $k^2 = k_x^2 + k_z^2 = \omega^2/a^2$, and let the distance between the boundaries of the cavity be $D = z_2 - z_1$. At z_1 and z_2 the vertical velocity must vanish, hence we have

$$w_1(z_1) = A_1 e^{i(\omega t - k_x x)} + A_2 e^{i(\omega t - k_x x)} = 0 \tag{54.35a}$$

and

$$w_1(z_2) = A_1 e^{i(\omega t - k_x x)} e^{-ik_z(z_2 - z_1)} + A_2 e^{i(\omega t - k_x x)} e^{ik_z(z_2 - z_1)} = 0. \tag{54.35b}$$

The first condition gives $A_2 = -A_1$. Thus $|A_1| = |A_2|$ and the phase shift between w^+ and w^- is 180°. The second condition gives

$$e^{-ik_zD} - e^{ik_zD} = -2i \sin (k_zD) = 0, \tag{54.36}$$

which implies that $k_z = n\pi/D$, $(n = 0, 1, 2, \ldots)$.

Because k_z is fixed, once ω and k_x are chosen (for a given a), (54.36) shows that standing waves exist only for certain combinations of k_x and ω, namely

$$k_x^2 = (\omega/a)^2 - (n\pi/D)^2. \tag{54.37}$$

Equation (54.37) shows that $n = 0$ corresponds to a horizontally propagating wave. For $n = 1$, $k_z = \pi/D$ and $\Lambda_z = 2D$, hence there is half a wavelength between z_1 and z_2. For $n = 2$ there is exactly one wavelength between z_1 and z_2. In general $\Lambda_z = 2D/n$ or $D = n\Lambda_z/2$, that is, the cavity is spanned by an integral number of half wavelengths.

If we allow the upper boundary to be open, so that $p_1(z_2) = 0$ while the lower boundary remains rigid, then $w_1(z_1) = 0$ again implies $A_1 = -A_2$, while at z_2

$$p_1(z_2) = -ik_zA_1e^{-ik_zD} + ik_zA_2e^{ik_zD} = 0 \tag{54.38}$$

implies

$$e^{-ik_zD} + e^{ik_zD} = 2 \cos (k_zD) = 0. \tag{54.39}$$

Therefore

$$k_z = (n + \tfrac{1}{2})\pi/D, \qquad (n = 0, 1, 2, \ldots), \tag{54.40}$$

which implies

$$\Lambda_z = 4D/(2n + 1) \tag{54.41}$$

or

$$D = (2n + 1)\Lambda_z/4. \tag{54.42}$$

That is, the cavity is spanned by an odd number of quarter wavelengths.

(*e*) *Wave Refraction in a Continuously Varying Medium* Now consider the propagation of an acoustic disturbance driven at frequency ω through a static medium in which the sound speed is a slowly varying function of spatial position, but is constant in time. The frequency of the wave remains unchanged as it propagates (because the oscillation is driven), but in general its amplitude and its wave vector will vary with spatial position.

In the limit that the wavelength of the disturbance is much smaller than the characteristic length over which the medium varies, we can use the language of *geometrical acoustics* (in analogy with geometrical optics) to describe the disturbance as a wave packet moving along a *ray*. The ray is the curve tangent to the propagation vector at each point in the medium, and is thus generated by the differential equation

$$(d\mathbf{x}/ds) = \mathbf{k}/k \equiv \mathbf{n}, \tag{54.43}$$

starting from initial conditions $\mathbf{k}=\mathbf{k}_0$ and $\mathbf{x}=\mathbf{x}_0$ at $s=s_0$. We clearly get a different ray, hence a different $\mathbf{x}(s)$ and $\mathbf{k}(s)$, for each choice of $\mathbf{k}_0$ at a fixed $\mathbf{x}_0$. Thus on a ray we must regard $\mathbf{x}$ and $\mathbf{k}$ as independent (canonical) variables.

Adopting this formalism we write $\omega=\omega(\mathbf{x}, \mathbf{k}, t)$. But ω is a constant of motion along the ray; therefore

$$\dot{\omega} \equiv (D\omega/Dt)_{\text{ray}} = (\partial\omega/\partial t) + \dot{\mathbf{x}} \cdot \nabla\omega + \dot{\mathbf{k}} \cdot \nabla_k\omega \equiv 0, \tag{54.44}$$

where ∇ denotes the gradient with respect to spatial coordinates holding $\mathbf{k}$ fixed, and ∇_k denotes the gradient with respect to wave-vector coordinates holding $\mathbf{x}$ fixed. For a driven wave in a static medium $(\partial\omega/\partial t)\equiv 0$. Furthermore, the velocity of the packet along the ray, $\dot{\mathbf{x}}$, is just the group velocity

$$\dot{\mathbf{x}} = \mathbf{v}_g = \nabla_k\omega = a\mathbf{n}. \tag{54.45}$$

Here we noted that for an acoustic wave $\omega = ak$, and that a depends on $\mathbf{x}$ but not $\mathbf{k}$. From (54.45) and (54.44) we conclude that

$$\dot{\mathbf{k}} = -\nabla\omega, \tag{54.46}$$

which for an acoustic wave yields

$$\dot{\mathbf{k}} = -k\nabla a. \tag{54.47}$$

Thus in a homogeneous medium $\mathbf{k}$ is constant along a ray, as expected. Alternatively

$$\dot{\mathbf{k}} \equiv D(k\mathbf{n})/Dt = k\dot{\mathbf{n}} + \dot{k}\mathbf{n} = k\dot{\mathbf{n}} - k(\dot{a}/a)\mathbf{n} \tag{54.48}$$

because $k=\omega/a$ and $\dot{\omega}\equiv 0$. But

$$\dot{a} = (\partial a/\partial t) + \dot{\mathbf{x}} \cdot \nabla a = a\mathbf{n} \cdot \nabla a \tag{54.49}$$

hence

$$\dot{\mathbf{n}} = -\nabla a + (\mathbf{n} \cdot \nabla a)\mathbf{n} \tag{54.50a}$$

or

$$(d\mathbf{n}/ds) = [-\nabla a + (\mathbf{n} \cdot \nabla a)\mathbf{n}]/a, \tag{54.50b}$$

and

$$\dot{k} = -k\mathbf{n} \cdot \nabla a. \tag{54.51}$$

Equations (54.50) and (54.51) show that if $\mathbf{n}$ lies along ∇a the direction of propagation $\mathbf{n}$ remains unchanged but the magnitude of $\mathbf{k}$ varies. On the other hand, if $\mathbf{n}$ is perpendicular to ∇a, then k is constant but $\mathbf{n}$ rotates away from the direction of ∇a. More generally, writing $G \equiv |\nabla a|$ and G_n for the component of ∇a along $\mathbf{n}$, we have

$$(\nabla a) \cdot d\mathbf{n} = -(ds/a)(G^2 - G_n^2) \leq 0. \tag{54.52}$$

Equation (54.52) shows that the change in the direction of $\mathbf{k}$ is always *away* from ∇a; that is, rays always refract away from regions of higher sound speed (i.e., higher temperature) toward regions of lower sound speed (i.e., lower temperature).

The change in wave amplitude can be found from a WKB analysis. For definiteness assume the properties of the medium to vary in z only, and write w_1 in terms of a constant amplitude and the phase functions $\phi(x, t)$ and $\psi(z)$:

$$w_1 = Ae^{i\phi(x,t)}e^{i\psi(z)}. \tag{54.53}$$

For steady-state wave motion in a static medium homogeneous in x, $\phi = \omega t - k_x x$. Pure acoustic waves satisfy the differential equation $(d^2w_1/dz^2) + k^2(z)w_1 = 0$. Acoustic-gravity waves satisfy

$$(d^2W/dz^2) + k^2(z)W = 0, \tag{54.54}$$

where $w_1 = W \exp(\int dz/2H)$. Using (54.53) in (54.54) we find

$$i\psi''(z) - [\psi'(z)]^2 + k^2(z) = 0, \tag{54.55}$$

where primes denote differentiation with respect to z.

A first approximation invoking the assumption of slow variation is obtained by setting $\psi''=0$, whence $\psi(z) \approx \pm\int k(z')\,dz'$. Using this value to estimate ψ'' in (54.55) we then have

$$[\psi'(z)]^2 \approx k^2(z) \pm ik'(z) \tag{54.56}$$

and thus

$$\psi(z) \approx \pm \int k(z')\{1 \pm i[k'(z')/k^2(z')]\}^{1/2}\,dz'. \tag{54.57}$$

Because we assume slow variations, $|k'(z)/k^2(z)| \ll 1$, hence $[1 \pm i(k'/k^2)]^{1/2} \approx 1 \pm i(k'/2k^2)$, thus

$$\psi(z) \approx \pm \int k(z')\,dz' + \tfrac{1}{2}i \ln k(z). \tag{54.58}$$

Using (54.58) in (54.53) we find

$$w_1(x, z, t) = \frac{A}{[k(z)]^{1/2}}\,e^{i(\omega t - k_x x)}e^{\pm i\int k(z')dz'} = \frac{A[a(z)]^{1/2}}{\omega^{1/2}}\,e^{i\phi}e^{\pm i\int k(z')dz'} \tag{54.59}$$

for acoustic waves, and

$$w_1 = W \exp\left(\int \frac{dz'}{2H}\right) = e^{\int dz'/2H}\,\frac{A}{[k(z)]^{1/2}}\,e^{i\phi}e^{\pm i\int k(z')dz'} \tag{54.60}$$

for acoustic-gravity waves.

(f) Wave Reflection in a Continuously Varying Medium We saw above that a pure acoustic wave is totally reflected at a discontinuity if the sound speed (i.e., temperature) in the second medium is high enough that $k_{zB}^2 = (\omega/a)^2 - k_x^2 \leq 0$. In a continuously varying medium, the ray path of a pure acoustic wave propagating into regions of ever-increasing temperature

continually bends away from the direction of ∇T until it turns around, that is, until the wave reflects. For a one-dimensional temperature variation this process is symmetric about ∇T (assumed to lie along the z axis), so if at some height z_0 the incoming wave has wave-vector components $[k_x, k_z(z_0)]$, the reflected wave at z_0 has components $[k_x, -k_z(z_0)]$.

In a stratified medium, wave reflection is governed by several effects. First consider gravity-modified acoustic waves for which the dominant terms in the dispersion relation (53.7) are obtained by ignoring buoyancy effects (equivalent to setting $\omega_g = 0$), which yields

$$k_z^2 = (\omega/a)^2 - (1/4H^2) - k_x^2 \qquad (54.61)$$

where H is the density scale height. Suppose first the wave is propagating into regions of increasing temperature. With increasing temperature, $(\omega/a)^2$ decreases as T^{-1} but $1/4H^2$ decreases as T^{-2}; thus at sufficiently high temperatures, we tend to recover the pure acoustic limit $k_z^2 = (\omega/a)^2 - k_x^2$ and reflection occurs as described in the preceding paragraph.

On the other hand, suppose a wave at height z_1, whose frequency is not much larger than the local value of the acoustic cutoff frequency $\omega_a(z_1)$, is propagating into regions of decreasing temperature. If, for the moment, we assume that $k_x^2 \ll (1/4H^2)$, then (54.61) simplifies to

$$k_z^2 = (\omega^2 - \omega_a^2)/a^2. \qquad (54.62)$$

Now ω_a rises as T^{-1}, hence it is clear that $k_z \rightarrow 0$, hence the wave is reflected at some height z_r where $\omega \approx \omega_a(z_r)$. Including the k_x^2 dependence from (54.61), we find that reflection occurs where

$$\omega = [\omega_a^2(z_r) + a^2 k_x^2]^{1/2}, \qquad (54.63)$$

that is, at a slightly smaller value of ω_a than given by (54.62). Reflection occurs when $\omega \rightarrow \omega_a$ because, as discussed in §53, the wave runs into an atmospheric resonance where it in effect tries to move the whole atmosphere simultaneously ($k_z = 0$), but is unable to overcome the inertia of all that material. More detailed analysis shows that as a wave propagates into regions of decreasing density scale height, p_1 lags farther and farther behind w_1 until they become 90° out of phase and the wave ceases to transport energy; beyond that point the wave appears only as an evanescent disturbance.

Reflection of internal gravity waves ($\omega \leq \omega_{BV}$) occurs for different reasons. Writing the dispersion relation as

$$k_z^2 = [(\omega_{BV}/\omega)^2 - 1]k_x^2 + (\omega^2 - \omega_a^2)/a^2 \qquad (54.64)$$

we see that the first term on the right-hand side is positive (and dominates for low-frequency waves with $\omega \ll \omega_{BV}$) while the second term is negative because $\omega_a > \omega_{BV}$ [cf. (53.10)]. Thus, as discussed in §53, for a fixed k_x, $k_z^2 \rightarrow 0$ as ω increases from very small values to the value set by the lower propagation boundary curve given by (53.12). The maximum frequency attained on this curve is ω_{BV}, which is reached asymptotically as $k_x \to \infty$. Thus in a varying medium, a gravity wave that can propagate with frequency ω at a given height is surely reflected at any height where the local value of $\omega_{BV}(z)$ decreases to ω. Physically this occurs because, as described in §52, ω_{BV} is the natural frequency for pure buoyancy oscillations in which buoyant fluid bobs vertically up and down at the effective free-fall rate. If one attempts to drive the oscillation faster, the fluid cannot fall back to its equilibrium position at the rate it is being driven, and the motion becomes purely evanescent in the vertical direction.

For small k_x, the lower propagation boundary is given by

$$\omega_0 = (\omega_{BV}/\omega_a)ak_x, \tag{54.65}$$

hence gravity waves can be reflected even when $\omega \ll \omega_{BV}$ if they propagate from a region where $\omega < \omega_0$ to one where $\omega = \omega_0$. Because the ratio (ω_{BV}/ω_a) is a slowly varying function (except in an ionizing medium, see below), this type of reflection tends to occur when gravity waves with small k_x propagate into regions of decreasing a.

We noted in §52 that $\omega_{BV}^2 > 0$ only in a stably stratified medium, and that gravity waves cannot exist in a convectively unstable region, where $\omega_{BV}^2 < 0$. It follows that near an interface between stably and unstably stratified regions, ω_{BV}^2 will fall to zero from the value characteristic of the stable region, hence *all* gravity waves will be reflected back into the stable layer, with only evanescent disturbances penetrating into the convective layer.

TEMPERATURE-GRADIENT AND IONIZATION EFFECTS ON THE BRUNT-VÄISÄIÄ FREQUENCY

For a perfect gas in hydrostatic equilibrium, $(d \ln \rho_0/dz) = (d \ln p_0/dz) - (d \ln T_0/dz)$, and $(dp_0/dz) = -\rho_0 g$; using these expressions in (52.6) we find

$$\omega_{BV}^2 = (\gamma - 1)(g/a)^2 + g(d \ln T_0/dz) = \omega_g^2(z) + g(d \ln T_0/dz). \tag{54.66}$$

In (54.66), $\omega_g(z)$ denotes the local value of the buoyancy frequency in the absence of a temperature gradient. We see that if the temperature increases upward ω_{BV} is increased over ω_g and the possibility of gravity-wave propagation is enhanced. If the temperature declines upward, ω_{BV} is smaller than ω_g, and gravity-wave propagation occurs only for a more restricted range of frequencies. Indeed, if (dT/dz) is sufficiently negative, $\omega_{BV}^2 < 0$ and gravity waves are completely suppressed, the atmosphere becoming convectively unstable.

For an ionizing gas we can rewrite (52.6) as

$$\omega_{BV}^2 = g[(1/H) - (1/\Gamma_1 H_p)] \tag{54.67}$$

where Γ_1 is given by (14.19) and H and H_p are, respectively, the density and pressure scale heights:

$$H^{-1} = -(d \ln \rho_0/dz) = H_p^{-1} + (d \ln T_0/dz) - (d \ln \mu/dz) \qquad (54.68)$$

and

$$H_p = -(d \ln p_0/dz)^{-1} = p_0/\rho_0 g = \mathcal{R}T/\mu g. \qquad (54.69)$$

Ionization effects must be accounted for in the mean molecular weight μ and in $(d \ln \mu/dz)$; for example, for pure hydrogen $\mu = 1/(1+x)$ [cf. (14.32)] where x is the degree of ionization [cf. (14.6)]. Note that the approach of Γ_1 toward unity in an ionization zone can cause the difference between $(1/H)$ and $(1/\Gamma_1 H_p)$ to become very small, thus sharply diminishing ω_{BV}^2 in that region.

An alternative expression for ω_{BV} can be obtained (**T3**) by expanding the density derivatives in (52.4) as $(d\rho/dz) = (\partial\rho/\partial p)_T (dp/dz) + (\partial\rho/\partial T)_p (dT/dz)$ and demanding that $(dp/dz)_{ad}$ inside the element equal (dp/dz) in the ambient atmosphere. One then finds

$$\omega_{BV}^2 = g\beta[(dT/dz)_{ad} - (dT/dz)_{at}], \qquad (54.70)$$

where, from (14.24) and (14.27),

$$\beta = -(\partial \ln \rho/\partial T)_p = T^{-1}\{1 + \tfrac{1}{2}x(1-x)[\tfrac{5}{2} + (\varepsilon_H/kT)]\} \qquad (54.71)$$

for ionizing hydrogen. Furthermore, we can evaluate $(dT/dz)_{ad}$ as

$$(dT/dz)_{ad} = (\partial T/\partial p)_s (dp/dz)_{ad} = (-\rho g)(T/p)(\Gamma_2 - 1)/\Gamma_2 = -(\Gamma_3 - 1)gT/a^2, \qquad (54.72)$$

where a^2 is given by (48.23). All thermodynamic quantities in (54.70) to (54.72) are to be evaluated allowing for ionization effects; for example Γ_1 and $(\Gamma_3 - 1)$ for ionizing hydrogen are given by (14.29) and (14.30).

In an ionizing medium, the acoustic cutoff frequency is again given by $\omega_a = a/2H$ as in (53.3), but now H is defined by (54.68) and a^2 by (48.23). If the gradients in T and μ are small enough to be neglected, then in an ionizing medium $H \approx H_p = a^2/\Gamma_1 g$, whence

$$\omega_{BV}^2 \approx (\Gamma_1 - 1)g^2/a^2. \qquad (54.73)$$

The ratio (ω_{BV}/ω_a) in (54.65) is then

$$(\omega_{BV}/\omega_a) \approx 2(\Gamma_1 - 1)^{1/2}/\Gamma_1, \qquad (54.74)$$

which varies with height only if Γ_1 varies.

FORMULATION OF THE WAVE EQUATION

To derive a wave equation for acoustic-gravity waves in a general stratified medium, it is convenient to work with scaled variables, as in §53. When the temperature, and therefore the density scale height H, varies with height, the ambient density is given by

$$\rho_0(z) = \rho_0(z_1) \exp\left[-\int_{z_1}^{z} dz'/H(z')\right] \tag{54.75}$$

where z_1 is an arbitrary reference height chosen at a convenient location. Thus defining

$$E(z) \equiv \exp\left[\int_{z_1}^{z} dz'/2H(z')\right] \tag{54.76}$$

we can write $\rho_0(z) = \rho_0(z_1)/E^2(z)$, which is the generalization of (53.16).

In scaling the perturbation variables, we note that k_z is no longer constant with height. We therefore absorb the dependence on k_z into *depth-dependent amplitude functions* defined by

$$\frac{\rho_1}{\rho_0 R(z)} = \frac{p_1}{p_0 P(z)} = \frac{T_1}{T_0 \Theta(z)} = \frac{u_1}{U(z)} = \frac{w_1}{W(z)} = e^{i(\omega t - k_x x)} E(z). \tag{54.77}$$

Using $i\omega U(z) = ik_x P(z)$ to eliminate $U(z)$ in favor of $P(z)$, the linearized fluid equations (52.24), (52.25), and (52.27) become

$$i\omega R(z) - (ik_x^2/\omega)P(z) - [(1/2H) - (d/dz)]W(z) = 0, \tag{54.78a}$$

$$gR(z) - [(1/2H) - (d/dz)]P(z) + i\omega W(z) = 0, \tag{54.78b}$$

and

$$i\omega R(z) - (i\omega/a^2)P(z) + \rho_0^{-1}[(d\rho_0/dz) - a^{-2}(dp_0/dz)]W(z) = 0. \tag{54.78c}$$

The coefficient of W in the energy equation (54.78c) reduces to

$$\begin{aligned} \rho_0^{-1}[(d\rho_0/dz) - a^{-2}(dp_0/dz)] &= -(1/H) + a^{-2}(p_0/\rho_0 H_p) \\ &= -[(1/H) - (1/\Gamma_1 H_p)] = -\omega_{BV}^2/g, \end{aligned} \tag{54.79}$$

hence (54.78c) can be rewritten as

$$i\omega R(z) - (i\omega/a^2)P(z) - (\omega_{BV}^2/g)W(z) = 0. \tag{54.78d}$$

The derivatives (dW/dz) and (dP/dz), which are $(-ik_z W)$ and $(-ik_z P)$ in an isothermal medium, now depend on gradients of T_0 and μ, and cannot be written in simpler form.

We eliminate R, first between (54.78a) and (54.78d), and then between (54.78b) and (54.78d) to obtain two equations relating P and W:

$$P(z) = [i\omega a^2/(\omega^2 - a^2 k_x^2)][(\omega_{BV}^2/g) - (1/2H) + (d/dz)]W(z) \tag{54.80a}$$

and

$$W(z) = [i\omega/(\omega^2 - \omega_{BV}^2)][(g/a^2) - (1/2H) + (d/dz)]P(z). \tag{54.80b}$$

From (54.78d) the density amplitude function is then

$$R(z)=[i\omega/(\omega^2-a^2k_x^2)][(\omega_{\rm BV}^2/g)(a^2k_x^2/\omega^2)-(1/2H)+(d/dz)]W(z). \tag{54.81}$$

The temperature perturbation for an ionizing gas can be obtained by first writing $(p_1/p_0)=(\partial\ln p_0/\partial\ln T_0)_\rho(T_1/T_0)+(\partial\ln p_0/\partial\ln\rho_0)_T(\rho_1/\rho_0)$, which is general, and then applying the cyclic relation $(\partial\ln p_0/\partial\ln\rho_0)_T/(\partial\ln p_0/\partial\ln T_0)_\rho=(\partial\ln T_0/\partial\ln\rho_0)_p$ to obtain

$$(T_1/T_0)=(\partial\ln T_0/\partial\ln\rho_0)_p[(\partial\ln\rho_0/\partial\ln p_0)_T(p_1/p_0)-(\rho_1/\rho_0)]. \tag{54.82}$$

Alternatively we can write $p=\rho kT/\mu m_H$, where μ is variable, which implies

$$(\partial\ln\rho_0/\partial\ln T_0)_p\equiv T\beta=1-(\partial\ln\mu/\partial\ln T_0)_p\equiv Q \tag{54.83}$$

[cf. (2.14) and (14.33)], and then using $(\partial\ln\rho_0/\partial\ln p_0)_T=p_0\kappa_T$ [cf. (2.15)] we have

$$T_1/T_0=[\kappa_T p_1-(\rho_1/\rho_0)]/Q \tag{54.84a}$$

or

$$\Theta(z)=[\kappa_T p_0 P(z)-R(z)]/Q. \tag{54.84b}$$

Substituting (54.80a) for P and (54.81) for R, and using (52.10) with T from (54.83) in the form

$$p_0\kappa_T\Gamma_1=Q(\Gamma_3-1)+1 \tag{54.85}$$

we obtain, after some reduction,

$$\Theta(z)=\frac{i\omega_{\rm BV}^2}{gQ\omega}W(z)+\frac{i\omega(\Gamma_3-1)}{(\omega^2-a^2k_x^2)}\left(\frac{\omega_{\rm BV}^2}{g}-\frac{1}{2H}+\frac{d}{dz}\right)W(z). \tag{54.86}$$

Recalling from (54.73) that $\omega_{\rm BV}^2$ is proportional to (Γ_1-1) when gradient terms can be neglected, we see that $\Theta(z)$ becomes small relative to $W(z)$ in ionization zones, where both Γ_1 and Γ_3 approach unity.

Because H and $\omega_{\rm BV}^2$ contain gradients of T and μ, we cannot simplify (54.80) and (54.81) as we did (53.32). Nevertheless (54.80a) and (54.80b) can be combined into a single wave equation for W (or P), namely

$$\begin{aligned}\frac{d^2W}{dz^2}&-\left[\frac{d}{dz}\ln\left(k_x^2-\frac{\omega^2}{a^2}\right)\right]\frac{dW}{dz}\\&+\left[\frac{\omega_{\rm BV}^2k_x^2}{\omega^2}+\frac{\omega^2}{a^2}-k_x^2-\frac{1}{4H^2}+\left(\frac{\omega_{\rm BV}^2}{g}-\frac{1}{2H}\right)\frac{d}{dz}\ln\left(k_x^2-\frac{\omega^2}{a^2}\right)\right.\\&\left.+\frac{d}{dz}\left(\frac{\omega_{\rm BV}^2}{g}-\frac{1}{2H}\right)\right]W=0.\end{aligned} \tag{54.87}$$

In an isothermal medium, (54.87) reduces to $(d^2W/dz^2)+k_z^2W=0$, where k_z^2 is given by (53.11) with ω_g^2 replaced by $\omega_{\rm BV}^2$ and a suitably general expression for ω_a. For a nonisothermal medium we can still obtain the same simple form by defining a new variable ψ as

$$\psi(z) = W(z)\{k_x^2 - [\omega/a(z)]^2\}/\{k_x^2 - [\omega/a(z_1)]^2\}, \qquad (54.88)$$

where z_1 is a convenient reference height. Let us also define

$$h_0(z) = [(\omega^2 - \omega_a^2)/a^2] + (\omega_{BV}^2 - \omega^2)(k_x^2/\omega^2); \qquad (54.89)$$

in the isothermal limit $h_0 = k_z^2$ and is constant. Then we find

$$[d^2\psi(z)/dz^2] + h(z)\psi(z) = 0 \qquad (54.90)$$

where

$$h(z) \equiv h_0(z) + \frac{d}{dz}\left(\frac{\omega_{BV}^2}{g} - \frac{1}{2H}\right) - \left(\frac{\omega_{BV}^2}{g} - \frac{1}{2H}\right)\frac{\omega^2}{a^2K^2}\frac{d \ln a^2}{dz}$$

$$- \left[\frac{3}{4}\left(\frac{\omega^2}{a^2K^2}\right)^2 + \frac{\omega^2}{2a^2K^2}\right]\left(\frac{d^2 \ln a^2}{dz^2}\right)^2 + \frac{\omega^2}{a^2K^2}\frac{d^2 \ln a^2}{dz^2}, \qquad (54.91)$$

with $K^2 \equiv k_x^2 - a^2\omega^2$.

SOLUTION OF THE WAVE EQUATION

We have reduced the linearized fluid equations (52.24) to (52.26) to a single, second-order, ordinary differential equation (54.90). This equation can be represented by a difference equation and solved as a two-point, boundary-value problem along lines discussed in §59. In practice, it is important to allow a variable step size Δz in the difference representation of $(d^2\psi/dz^2)$ because the vertical wavelength $2\pi/[h(z)]^{1/2}$ varies substantially over distances comparable to a wavelength, especially for internal gravity waves.

At the lower boundary, we specify a velocity or pressure perturbation that drives the wave or else specify the net upward energy flux; because the problem is linear these conditions are all essentially equivalent. At the upper boundary, the easiest condition to impose is to allow only an outgoing wave. The justification for this condition is that in a stratified atmosphere the velocity amplitudes increase exponentially with height, and we can always place the upper boundary above the height range where we would expect the waves (in a nonlinear treatment) to become highly nonlinear and dissipate, and thus be unable to reflect back into the region of interest.

From the (complex) solution for $\psi(z)$, $W(z)$ can be determined from (54.88) and (dW/dz) can then be obtained from (54.87). The scaled amplitude functions $P(z)$, $R(z)$, and $\Theta(z)$ follow from (54.80a), (54.81), and (54.86). The linear wave perturbations ρ_1, p_1, T_1, u_1, and w_1 as functions of (x, z, t) are then determined from (54.77). All the perturbations are complex variables; we take the physical perturbation to be the real part of the corresponding perturbation variable. Thus at point (x_i, z_i, t_i) in the fluid, the velocity is

$$\mathbf{v}_1(x_i, z_i, t_i) = \{\text{Re}\,[u_1(x_i, z_i, t_i)], 0, \text{Re}\,[w_1(x_i, z_i, t_i)]\}, \tag{54.92}$$

the gas pressure is

$$p(x_i, z_i, t_i) = p_0(z_i) + \text{Re}\,[p_1(x_i, z_i, t_i)], \tag{54.93}$$

and similarly for the other variables.

The magnitudes and phases of the wave perturbations are obtained from the standard formulae for complex variables. From (54.77) one sees that the phase lag between any two variables (say p_1 and w_1) is the same as that between their amplitude functions [i.e., $P(z)$ and $W(z)$]. Moreover one sees that although the phase of each variable changes with x and t, the phase *lag* between two variables is a function of z only. Thus

$$\delta_{PW}(z_i) = \delta_P(x_i, z_i, t_i) - \delta_W(x_i, z_i, t_i) \equiv \delta_P(0, z_i, 0) - \delta_W(0, z_i, 0). \tag{54.94}$$

Because the derivative (dW/dz) in (54.80), (54.81), and (54.86) cannot in general be replaced by $-ik_zW$ we cannot write analytic expressions like (53.32) for the ratios (P/W), (R/W), and (Θ/W) or like (53.37) for phase differences. Instead, these must be computed from the numerical solutions for the amplitude functions. However, in the special case that k_z and H are almost constant, so that we can take $(dW/dz) \approx -ik_zW$ and $H \approx H_p$, we find the phase differences for an ionizing gas are

$$\tan \delta_{PW} \approx (2k_zH)^{-1}[(\Gamma_1 - 2)/\Gamma_1], \tag{54.95a}$$

$$\tan \delta_{RW} \approx (2k_zH)^{-1}\{[2(\Gamma_1 - 1)a^2k_x^2/\Gamma_1\omega^2] - 1\}, \tag{54.95b}$$

and

$$\tan \delta_{\Theta W} \approx (2k_zH)^{-1}[(\Gamma_1 - 2)/\Gamma_1] + (k_zH)^{-1}[1 - (ak_x/\omega)^2][(\Gamma_1 - 1)/\Gamma_1(\Gamma_3 - 1)Q]. \tag{54.95c}$$

STRUCTURE OF THE SOLAR ATMOSPHERE

To illustrate the theory developed above, we discuss the propagation of linear acoustic-gravity waves in the solar atmosphere using a semiempirical model derived from an analysis of spectral data (**V2**), (**V3**). The model is approximately in hydrostatic equilibrium, but small adjustments are necessary to match observed scale heights. The required nongravitational forces are usually *parameterized* in terms of a "turbulent pressure" gradient; the ultimate origin of these forces is presently unknown but presumably they result from small-scale fluid flow and magnetic fields.

The temperature structure, shown in Figure 54.1, exhibits an initial decline as implied by radiative equilibrium at an open boundary (cf. §82). This region, known as the *photosphere*, contains the surface (about one photon mean free path into the Sun) from which most of the visible light is emitted. Moreover, this is the region where radiation interacts strongly

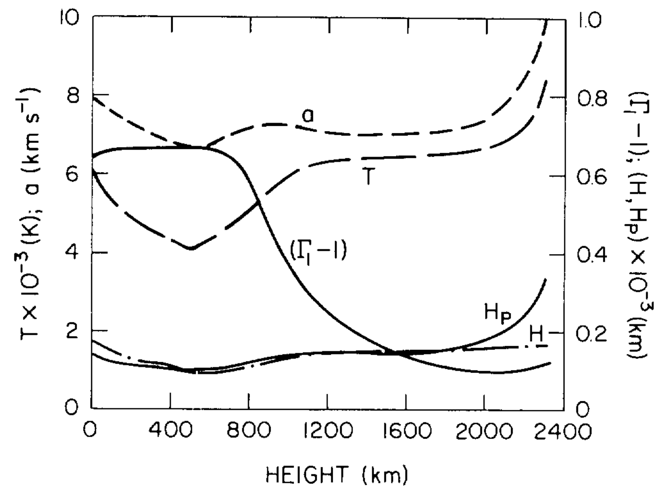

Fig. 54.1 Atmospheric structural parameters in a model solar atmosphere.

with wave-induced local temperature perturbations (cf. §102). At about 500 km above the visible surface, the temperature passes through the *temperature minimum region* and then rises outward in the *chromosphere* where the cores of strong spectral lines in the solar spectrum are formed. The temperature rise is thought to result from dissipation of mechanical (i.e., wave) and magnetic energy. The initial rise is followed by a plateau from about 1000 to 2000 km; here nonradiative energy input continues to increase the internal energy of the gas, but nearly all this energy is consumed in ionizing hydrogen, and the temperature rises only slightly. Above the plateau, the temperature rises abruptly through the *transition region* (whose thermal structure is determined by a balance between radiative losses, nonradiative energy dissipation, and thermal conduction) into the *corona*, a tenuous envelope at about 1.5×10^6 K, which is the seat of the *solar wind* (cf. §§61 and 62).

The propagation of acoustic-gravity waves, including their refraction and reflection properties, is governed by the average values and gradients of the temperature, mean molecular weight, and ionization fraction, which determine Γ_1, Γ_2, and Γ_3, H and H_p, and the parameters a^2, ω_a^2, and ω_{BV}^2 that appear in the wave equation and dispersion relation.

As shown in Figure 54.1, the adiabatic exponents are all near $\frac{5}{3}$ in the photosphere and begin to decrease a short distance above the temperature minimum, as hydrogen begins to ionize. They continue to decrease until the top of the temperature plateau near 2200 km, at which point they rise sharply back toward $\frac{5}{3}$ as hydrogen ionization becomes essentially complete.

The sound speed, also shown in Figure 54.1, exhibits relatively little variation over the height range 0 to 2000 km; a varies only as $T^{1/2}$ and

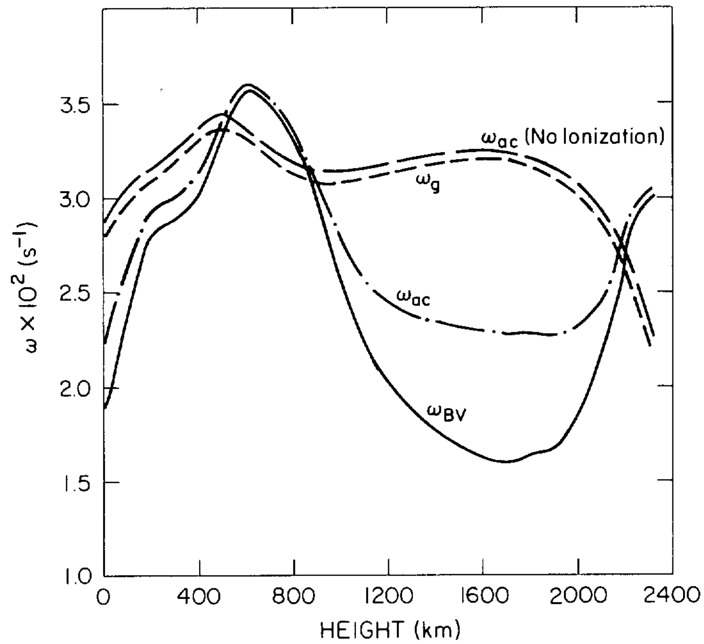

Fig. 54.2 *Upper curves:* acoustic cutoff frequency and buoyancy frequency in a model solar atmosphere, assuming $\Gamma_1 \equiv \frac{5}{3}$. *Lower curves:* acoustic cutoff frequency and Brunt–Väisälä frequency in a model solar atmosphere, allowing for ionization and temperature gradient effects.

even this variation is largely offset by the decrease of Γ_1 in the chromosphere. In the transition region a rises sharply as T increases to coronal values. The acoustic cutoff frequency $\omega_a = a/2H$, shown in Figure 54.2, has a distinct maximum near 750 km where H has a definite minimum and a has a weaker maximum. Above that height ω_a decreases in the chromosphere because H increases as T increases and μ decreases [cf. (54.68)] while a decreases slightly owing to the decrease in Γ_1. In the transition region ω_a decreases sharply $\propto T^{-1/2}$ as T rises to coronal values.

The Brunt-Väisälä frequency exhibits much more dramatic changes, primarily as the result of changes in Γ_1. From Figure 54.1 one sees that H and H_p are nearly equal in most of the chromosphere, hence in this region $\omega_{BV}^2 \approx (\Gamma_1 - 1)g/\Gamma_1 H$ [cf. (54.67)]. Noting that H varies relatively slowly, one infers that ω_{BV} in the chromosphere should respond mainly to changes in $(\Gamma_1 - 1)/\Gamma_1$. The correctness of this inference is seen from Figure 54.2, which shows ω_{BV} calculated with realistic values of Γ_1 and with $\Gamma_1 \equiv \frac{5}{3}$; almost all of the chromospheric drop in ω_{BV} is caused by ionization effects. As hydrogen becomes fully ionized in the upper chromosphere, ω_{BV} rises slightly again and then decreases sharply to very small values in the corona in response to extremely high coronal temperatures. In addition, ω_{BV}

decreases from its maximum near 750 km downward into the photosphere. Indeed, just below the photosphere there is a convection zone that extends deep into the solar envelope; here $\omega_{BV}^2<0$, hence ω_{BV} must pass through zero near the bottom of the photosphere.

TEMPERATURE-GRADIENT AND IONIZATION EFFECTS ON THE DIAGNOSTIC DIAGRAM

Figure 54.3 shows propagation boundary curves in the diagnostic diagram for four heights in the model solar atmosphere: (1) in the low photosphere; (2) at 750 km, where ω_a has its maximum value of about $3.4\times10^{-2}\,s^{-1}$ ($\sim$185 s period); (3) at 1700 km where ω_{BV} has a pronounced minimum at about $1.2\times10^{-2}\,s^{-1}$ ($\sim$525 s period); and (4) in a 1.5×10^6 K corona of fully ionized hydrogen and helium. Full allowance is made for temperature gradients and ionization effects.

The two shaded areas on the diagram indicate ranges of (k_x, ω) for which acoustic-gravity waves can be trapped in some region of the solar atmosphere. Region I corresponds to acoustic waves trapped in the *chromospheric cavity* that extends from about 750 km height to the transition region or corona. Acoustic waves with $\omega>1.6\times10^{-2}\,s^{-1}$ can propagate in the middle to upper chromosphere; those propagating downward are reflected back up if $\omega\leq3.4\times10^{-2}\,s^{-1}$, while those propagating upward refract away from the steep temperature rise to the corona, and totally reflect if they have horizontal wavenumbers $k_x>10^{-4}\,km^{-1}$ ($\Lambda_x\leq63{,}000$ km). Note also that acoustic waves with ω slightly less than

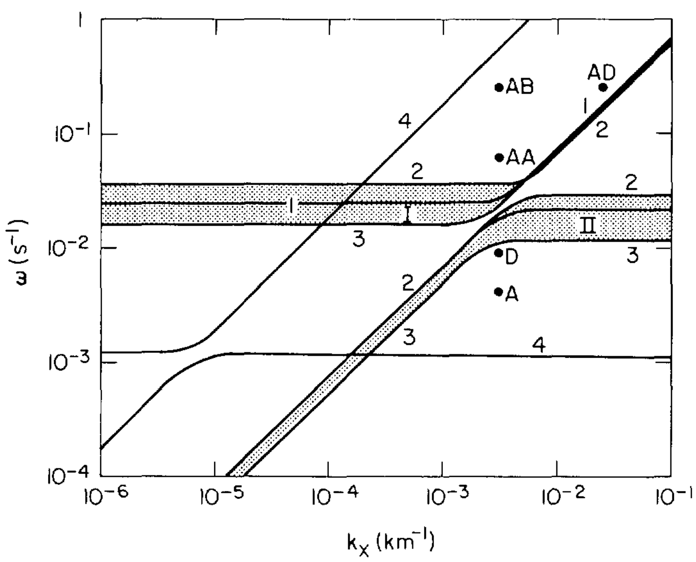

Fig. 54.3 Propagation boundary curves at four heights in a model solar atmosphere. Shaded areas indicate ranges of (k_x, ω) in which waves may be trapped in a cavity. Lettered dots mark (k_x, ω) values for representative waves discussed in text.

$\omega_{a,\max}$ can propagate freely both just above and just below the layer at 750 km where $\omega_a = \omega_{a,\max}$. Thus acoustic waves at these frequencies propagating upward from the subphotospheric convection zone tunnel through a thin layer around 750 km as evanescent waves and penetrate into the chromospheric cavity, where they are trapped, with most of their energy remaining.

Internal gravity waves can be trapped similarly in the photosphere and low-chromosphere region for the (k_x, ω) values shown as region II in Figure 54.3. As mentioned above, ω_{BV} rapidly decreases toward zero near the bottom of the photosphere. Hence, downward propagating gravity waves will be reflected upward at the interface between the stably stratified photosphere and the convection zone. Gravity waves propagating upward in the photosphere having frequencies greater than about $1.2\times10^{-2}\,s^{-1}$ will be reflected downward from the middle chromosphere where ω_{BV} drops to a local minimum. Gravity waves with $\omega < 1.2\times10^{-2}\,s^{-1}$ and wavenumbers in the small-k_x end of the propagation domain are likewise reflected downward from the chromosphere. Gravity waves with ω only slightly above $1.2\times10^{-2}\,s^{-1}$ can tunnel into the upper chromosphere, where they propagate until reflected by the coronal temperature rise. In the corona only gravity waves with periods greater than about two hours can propagate.

ADIABATIC ACOUSTIC-GRAVITY WAVES IN THE SOLAR ATMOSPHERE

Results obtained from numerical solutions of (54.90) in the model solar atmosphere just described exhibit the refraction, partial reflection, and tunneling effects described earlier in this section. They also demonstrate the general trends for phase differences and amplitude ratios of high- and low-frequency wave perturbations as discussed in §53 [cf. (53.33) and (53.37)]. The waves chosen as representative examples are shown as lettered dots in Figure 54.3. They comprise sets of freely propagating gravity waves and acoustic waves, some of which have very small k_z somewhere in the computational domain, plus one case of a fully reflected gravity wave that tunnels through a thin evanescent layer. More extensive results are given in (**M3**) and (**M4**).

One readily sees a strong response to the dominant features of the solar model in the computed height dependence of the eigenfunctions, phase differences, and relative amplitudes of the waves. Phase lags are shown in Figure 54.4 for two of the freely propagating gravity waves, for the reflected gravity wave, and for an acoustic wave with relatively small k_z (long period). The phase lags discriminate readily between the acoustic and gravity-wave regimes and exhibit effects of partial reflection more strikingly than do the other wave properties. From the discussion following (53.73), we expect all the perturbations to be approximately in phase for acoustic waves, with the largest phase differences occurring where ω_a/ω is largest; the magnitudes of the phase differences in Figure 54.4d do in fact

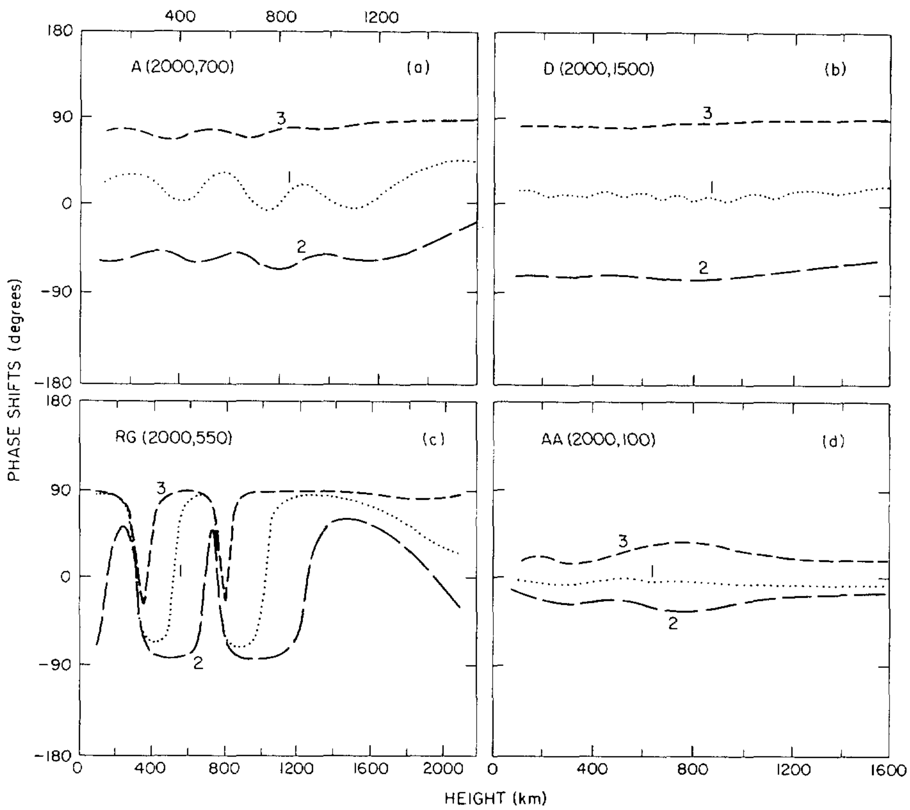

Fig. 54.4 Phase shifts for acoustic-gravity waves in a model solar atmosphere. (1) δ_{PW}. (2) δ_{RW}. (3) δ_{TW}.

follow closely the variations in ω_a shown in Figure 54.2. Also, as expected from (53.37), p_1 and ρ_1 lag w_1 for the acoustic wave, and T_1 leads. The example shown in Figure 54.4d is a relatively low-frequency acoustic wave; at higher frequencies the phase differences are much smaller, as can be seen in the eigenfunctions for a high-frequency acoustic wave shown in Figure 54.5d.

Gravity waves, which experience substantial partial reflection from the large variation of ω_{BV} in the chromosphere, show oscillations in the phase differences on a scale of half the vertical wavelength. These oscillations are an interference pattern that results from the superposition of upward and downward propagating waves. Figures 54.4a,b,c show phase differences for gravity waves that propagate energy upward, hence have negative k_z and propagate phase downward. In this case p_1 slightly leads w_1, while T_1 leads and ρ_1 lags by amounts that approach 90° as ω/ω_{BV} and ω/ak_x become very small. The phase oscillations approach $\pm 90°$ as reflection becomes nearly complete, which accounts for the extreme behavior seen in Figure 54.4c for a gravity wave that is evanescent between about 1525 and 1750 km in height.

The eigenfunctions for the same three gravity waves and for a high-frequency acoustic wave are shown in Figure 54.5; the relative perturbation amplitudes for this set of waves are shown in Figure 54.6. Here we see other effects of the variation of temperature and ionization in the solar model. The eigenfunctions again show the marked difference in phase behavior between gravity (a, b, c) and acoustic (d) waves.

Figure 54.6d shows that acoustic waves in the solar atmosphere behave very nearly in accordance with the asymptotic (isothermal) relations (53.34). Here $k_z \approx k$, hence $\sin\alpha \approx 1$ and $\cos\alpha$ is very small. One sees that $|u_1|/a \approx (|w_1|/a)\cos\alpha$ is indeed small, and that $|p_1|/p_0 \approx \Gamma|w_1|/a$ and $|T_1|/T_0 \approx (\Gamma-1)^{1/2}|w_1|/a$ both appear to have the perfect-gas value of Γ at

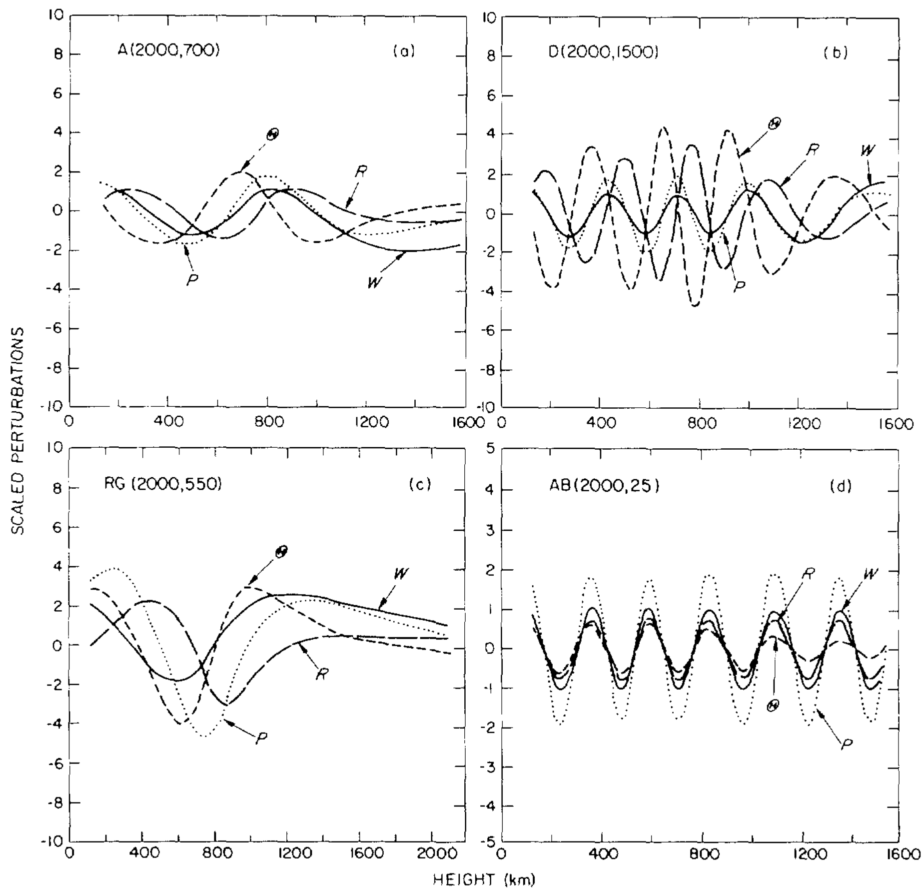

Fig. 54.5 Scaled eigenfunctions for acoustic-gravity waves in a model solar atmosphere.

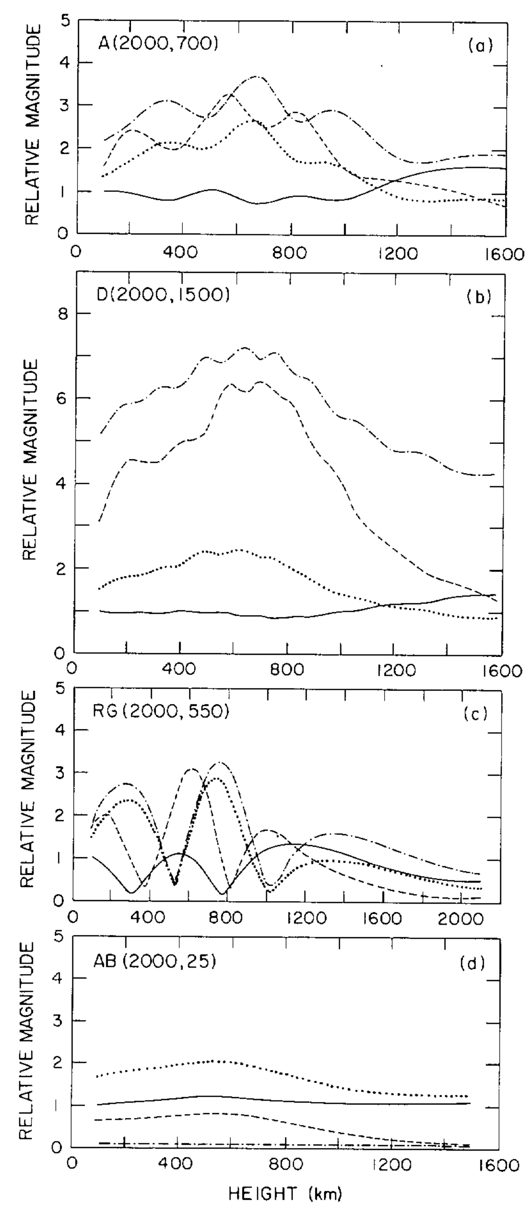

Fig. 54.6 Relative amplitudes for acoustic-gravity waves in a model solar atmosphere. *Solid curves: W. Dot-dash curves: U. Dotted curves: P. Dashed curves:* Θ.

low heights, and to vary in the manner predicted by (53.34) as Γ decreases nearly to unity with increasing height in the chromosphere.

Similarly the amplitude ratios for the gravity waves in Figure 54.6a and 54.6b follow closely the relations (53.35). The ratios $|T_1|/T_0$ and $|u_1|/a$, both of which are proportional to (ω_{BV}/ω), are much larger for the lower-frequency wave D than for the higher-frequency wave A, and both follow the increase in ω_{BV} in the low chromosphere. The ratio $|T_1|/T_0$ drops sharply below $|u_1|/a$ in the middle chromosphere because Γ approaches unity. The ratio $|\rho_1|/\rho_0$, not shown in Figure 54.6, follows $|T_1|/T_0$ very closely for gravity waves; the physical reason is that in gravity waves the density perturbation is produced mainly by the difference in temperature between the adiabatically oscillating fluid and its surroundings. Figure 54.6 contains only gravity waves of a single horizontal wavelength, hence it does not show the dependence of $|p_1|/p_0$ on k_x^{-1} which results from the fact that in gravity waves the pressure perturbation acts mainly to drive horizontal flow; hence as the horizontal extent of the flow increases (Λ_x becomes larger) the pressure perturbation needed to drive it increases.

Figure 54.6c illustrates in another way the characteristics of standing waves: for a perfect standing wave the nodes of $|p_1|$, $|\rho_1|$, and $|T_1|$ would all

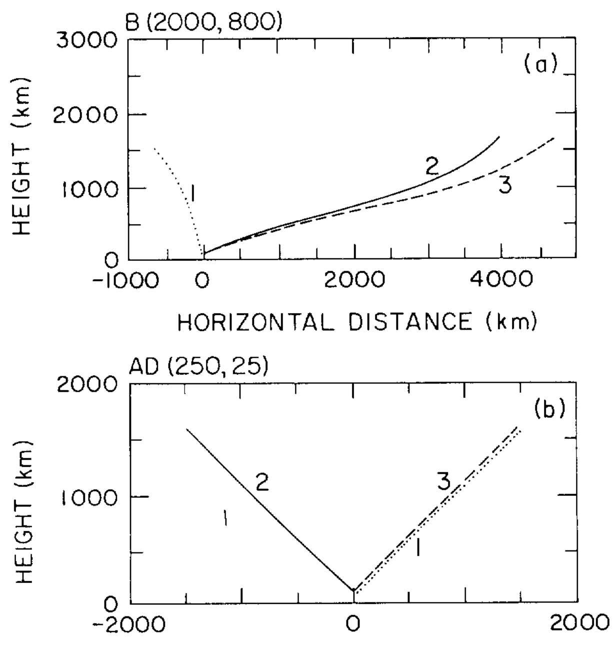

Fig. 54.7 Constant phase path (curve 1), phase-velocity path (curve 2), and group-velocity path (curve 3) for acoustic-gravity waves in a model solar atmosphere.

fall midway between the nodes of $|w_1|$, with two nodes per vertical wavelength, and the amplitude would be exactly zero at each node.

Figure 54.7 illustrates refraction of acoustic-gravity waves in response to the continuous variation of properties of the solar atmosphere. Because the sound speed changes little with height, high-frequency waves show little bending of the direction of the phase velocity $\mathbf{v}_p$ or group velocity $\mathbf{v}_g$. Gravity waves, in contrast, show strong refraction from the large changes in ω_{BV}, with $\mathbf{v}_p$ bending away from the vertical as ω_{BV} decreases, and $\mathbf{v}_g$ bending toward the vertical. Though the direction of $\mathbf{v}_g$ tends toward the vertical, the magnitudes of both u_g and w_g decrease to zero as ω_{BV} decreases to ω.

5.3 Shock Waves

The theory developed in §§5.1 and 5.2 applies only to small-amplitude disturbances, which propagate essentially adiabatically and are damped only slowly by dissipative processes. As the wave amplitude increases, this simple picture breaks down because of the effects of the *nonlinear* terms in the equations of hydrodynamics. When nonlinear phenomena become important, the character of the flow alters markedly. In particular, in an acoustic disturbance a region of compression tends to overrun a rarefaction that precedes it; thus as an acoustic wave propagates, the leading part of the profile progressively steepens, eventually becoming a near discontinuity, which we identify as a *shock*.

Once a shock forms it moves through the fluid supersonically and therefore outruns preshock acoustic disturbances by which adjustments in local fluid properties might otherwise take place; it can therefore persist as a distinct entity in the flow until it is damped by dissipative mechanisms. The material behind a shock is hotter, denser, and has a higher pressure and entropy than the material in front of it; the *stronger* the shock (i.e., the higher its velocity) the more pronounced is the change in material properties across the discontinuity. The rise in entropy across a shock front implies that wave energy has been dissipated irreversibly; this process damps, and ultimately destroys, the propagating shock (sometimes rapidly).

In contrast to acoustic waves, internal gravity waves do not develop shocks. Instead in the nonlinear regime they *break* and degenerate into turbulence. We will not discuss these phenomena in this book; see for example, (**M3**) and (**M4**).

Shock phenomena are of tremendous importance in astrophysics. As we saw in §5.2, the growth of waves to finite amplitude occurs naturally and inevitably in an atmosphere having an exponential density falloff. Thus, as Biermann (**B3**), (**B4**) and Schwarzschild (**S8**) first recognized, small-amplitude acoustic disturbances generated by turbulence in a stellar convection zone can propagate outward with ever-increasing amplitude until they steepen into shocks that dissipate their energy, thus heating the